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NONLINEAR COUPLING OF
HIGH FREQUENCY ELECTROMAGNETIC WAVES
IN BOUNDED PLASMAS

by

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A THESIS
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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Nonlinear Coupling of High Frequency Electromagnetic Waves in Bounded Plasmas" submitted by Lance A. Turlock in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

A method is presented for the solution of second order electromagnetic fields in a bounded plasma which is excited by two high frequency electromagnetic waves. In particular, the problem considered is the optimization of second order fields in a uniform plasma confined to a circular guide. Consideration is also given to the amount of ion heating that may be expected through the wave mixing process when a static magnetic field is applied to the plasma. Two underlying assumptions in the analysis are: (1) the frequencies of the incident waves are sufficiently high that they achieve a thorough, uniform penetration of the plasma; (2) the thermal velocities of the species comprising the plasma are much less than the applicable phase velocities of any disturbances; the latter assumption justifying a cold plasma model.

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LIST OF SYMBOLS

m_e	electron mass
m_i	ion mass
m_r	m_e/m_i
n_o	number density of particles
e	electronic charge
K	Boltzmann's constant
T	absolute temperature
c	velocity of light
ρ	charge density
α	When used as a subscript in an equation, α denotes that the equation is written for each charge species.
q_α	charge of the α-species
$J_n(x)$	a Bessel function of order n
$\langle x \rangle$	denotes average value of x

A vector, V, is denoted by $\underline{V}$.

A matrix, M, is denoted by $[\overline{M}]$.

The transpose of $[\overline{M}]$ is denoted by $[\overline{M}]^T$.

CHAPTER 1 INTRODUCTION

1.1 General

In recent years, the problems associated with the nonlinear interaction of high frequency electromagnetic waves in a plasma have drawn much attention. It has been found in a number of theoretical papers that it is possible to achieve a resonant mixing of two waves of frequencies ω_1 and ω_2 propagating in an unmagnetized plasma when the difference frequency, $\omega_1 - \omega_2$, is approximately equal to the electron-plasma frequency $\omega_{pe} = [4\pi e^2 n_o/m_e]^{1/2}$. The mixing process can result in appreciable second order fields.

In 1964, Salat and Schluter (1), described a diagnostic procedure whereby plasma density could be determined by measuring the intensity of light scattered by a plasma excited by a single laser beam. The scattered light rays from different points in the scattering volume were generally out of phase except in a particular scattering direction. By measuring the angle between the incident signal and the direction in which light was scattered in phase at the harmonic frequency $2\omega_1$, the plasma density could be determined. A major disadvantage, however, was that the amount of energy scattered was extremely small in relation to the source energy.

A modification of the above procedure, using two laser beams, results in a much more favourable scattered

to incident energy ratio. Such a method was considered by Kroll et al (2). By mixing two laser beams, it was found that considerable enhancement of the scattering cross-section was achieved when the difference frequency $\omega_1 - \omega_2 \simeq \omega_{pe}$, giving a significant amplitude to the scattered wave. A comparison was drawn for a specific example showing that the level of the scattered signal using a single laser, as compared to using two lasers, was $\simeq 10^{-16}$. A more detailed account of this procedure is given by Shkarofsky (3).

A further modification has been suggested by introducing a third laser beam. A disadvantage of the above methods is that the detectors must look almost directly into the incident beams because the angle between the scattered light and the laser beams is quite small. Salat (4) has shown that scattering a third beam off the fluctuations due to the first two can lessen this problem considerably, however the orientation of the third beam is subject to high tolerances.

The above diagnostic procedures have involved the use of laser beams and treated the plasma as an infinite medium. It has recently been found by Stern and Tzoar (5), that two microwave signals will stimulate oscillations in a plasma at electron densities at which the plasma can support resonant modes (for a uniform plasma, this is the electron-plasma frequency). With this type of problem, where the incident waves mix throughout the plasma and are a function of the confining structure, the plasma must be considered

as finite to achieve a correct interpretation of experimental results. In this thesis, a theoretical approach to the problem of wave interaction in a finite plasma is presented. Initially, a treatment of an unmagnetized plasma will be given, followed by a similar problem when a static magnetic field is applied in the direction of propagation of the incident waves. With respect to the latter case, it has been suggested by James and Thompson (6), that a possibility exists for significant ion heating through a wave mixing process. However, their theory was based on an infinite plasma excited by plane waves. To obtain a more accurate picture of what is achievable, the plasma must be considered as finite, since the degree of coupling will have a dependence on the shape of the confining structure. Hence a major consideration for the magnetized plasma will be an investigation into the degree of ion heating that might be expected in a finite structure.

1.2 Plasma Model

In this thesis the cold plasma model will be used. It will be assumed that each charge species (ions and electrons) obeys an equation of continuity:

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot(n_\alpha \underline{v}_\alpha) = 0 \qquad \ldots(1.1)$$

and an equation of motion:

$$\frac{\partial \underline{v}_\alpha}{\partial t} + (\underline{v}_\alpha \cdot \nabla)\underline{v}_\alpha = \frac{q_\alpha}{m_\alpha}\left(\underline{E} + \frac{\underline{v}_\alpha \times \underline{B}}{c}\right)$$

$$+ \sum_\alpha (\text{collision terms})_\alpha \quad \ldots(1.2)$$

Terms involving collisions will be considered later, depending on the particular problem being investigated.

In an unmagnetized plasma, thermal effects can be neglected provided (7) the thermal velocities, $v_{\theta\alpha}$, of the ions and electrons are much less than the phase velocities of any disturbances.

$$v_{\theta\alpha} = \sqrt{\frac{KT}{m_\alpha}} << \frac{\omega}{k} \qquad \ldots(1.3)$$

where ω and k are the frequency and wavenumber of the disturbance. The equivalent condition for a magnetized plasma is

$$\sqrt{\frac{KT}{m_\alpha}} << \frac{\omega \pm \Omega_\alpha}{k} \qquad \ldots(1.4)$$

where Ω_α is the cyclotron frequency of the α-species. Further, it has been verified (8), when (1.3) and (1.4) are satisfied, that to second order the density and current fluctuations produced in a plasma by external waves are the same, whether derived from kinetic theory or cold plasma theory. It follows that the resulting fields must necessarily be the same from either approach. Since cold plasma theory is mathematically less involved than kinetic theory,

this approach has been used.

1.3 General Equations

Assume two electromagnetic waves of frequencies ω_1 and ω_2 are incident on a uniform plasma confined to a cylindrical waveguide of circular cross-section. In addition, assume that ω_1 and ω_2 are sufficiently high that shielding of these waves is negligible to first order - there is essentially an impedance match between free space and the plasma. Hence, no reflection of the incident waves occurs and the waves are not attenuated in the plasma. Under this assumption, the first order fields are the solutions of Maxwell's equations in an empty waveguide.

To consider a particular problem, assume that the incident waves are transverse electric, TE_{0m} and TE_{0n}, modes. Hence, from Maxwell's equations:

$$\nabla \cdot \underline{E} = 4\pi\rho \qquad \ldots(1.5)$$

$$\nabla \cdot \underline{B} = 0 \qquad \ldots(1.6)$$

$$\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \qquad \ldots(1.7)$$

$$\nabla \times \underline{B} = \frac{4\pi}{c}\,\underline{j} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t} \qquad \ldots(1.8)$$

and using cylindrical co-ordinates (r,ϕ,z), the solutions for the incident fields are:

(a) TE_{0m}

$$\left.\begin{aligned} B_z^1 &= \bar{B}_z^1 J_0(k_{11}r)\cos(\omega_1 t+k_{21}z+\alpha_1) \\ E_\phi^1 &= \bar{E}_\phi^1 J_1(k_{11}r)\sin(\omega_1 t+k_{21}z+\alpha_1) \\ B_r^1 &= \bar{B}_r^1 J_1(k_{11}r)\sin(\omega_1 t+k_{21}z+\alpha_1) \end{aligned}\right\}\quad(1.9)$$

where

$$k_{11} = \frac{s_{1m}}{r_o} \qquad \ldots(1.10)$$

with s_{1m} being the m^{th} root of $J_1(s) = 0$, and r_o the radius of the waveguide. Also

$$\left(\frac{\omega_1}{c}\right)^2 = k_{11}^2 + k_{21}^2 \qquad \ldots(1.11)$$

$$\bar{B}_z^1 = \frac{k_{11}c}{\omega_1}\bar{E}_\phi^1, \quad \bar{B}_r^1 = \frac{k_{21}c}{\omega_1}\bar{E}_\phi^1 \qquad \ldots(1.12)$$

(b) TE_{0n}

$$\left.\begin{aligned} B_z^2 &= \bar{B}_z^2 J_0(k_{12}r)\cos(\omega_2 t+k_{22}z+\alpha_2) \\ E_\phi^2 &= \bar{E}_\phi^2 J_1(k_{12}r)\sin(\omega_2 t+k_{22}z+\alpha_2) \\ B_r^2 &= \bar{B}_r^2 J_1(k_{12}r)\sin(\omega_2 t+k_{22}z+\alpha_2) \end{aligned}\right\}\quad(1.13)$$

where

$$k_{12} = \frac{s_{1n}}{r_o} \qquad \ldots(1.14)$$

with s_{1n} being the n^{th} root of $J_1(s) = 0$. Also

$$\left(\frac{\omega_2}{c}\right)^2 = k_{12}^2 + k_{22}^2 \qquad \ldots(1.15)$$

$$\bar{B}_z^2 = \frac{k_{12}c}{\omega_2}\bar{E}_\phi^2, \quad \bar{B}_r^2 = \frac{k_{22}c}{\omega_2}\bar{E}_\phi^2 \qquad \ldots(1.16)$$

The fields are now expressed in exponential form:

$$\begin{bmatrix} B_z^1 \\ E_\phi^1 \\ B_r^1 \end{bmatrix} = e^{i(\omega_1 t+k_{21}z)} \begin{bmatrix} \hat{B}_z^1 \\ \hat{E}_\phi^1 \\ \hat{B}_r^1 \end{bmatrix} + e^{-i(\omega_1 t+k_{21}z)} \begin{bmatrix} \hat{B}_z^{1*} \\ \hat{E}_\phi^{1*} \\ \hat{B}_r^{1*} \end{bmatrix} \qquad \ldots(1.17)$$

where

$$\left.\begin{aligned} \hat{B}_z^1 &= \frac{1}{2}\,\bar{B}_z^1 J_0(k_{11}r)e^{i\alpha_1} \\ \hat{E}_\phi^1 &= \frac{1}{2i}\,\bar{E}_\phi^1 J_1(k_{11}r)e^{i\alpha_1} \\ \hat{B}_r^1 &= \frac{1}{2i}\,\bar{B}_r^1 J_1(k_{11}r)e^{i\alpha_1} \end{aligned}\right\}(1.18)$$

$$\begin{bmatrix} B_z^2 \\ E_\phi^2 \\ B_r^2 \end{bmatrix} = e^{i(\omega_2 t+k_{22}z)} \begin{bmatrix} \hat{B}_z^2 \\ \hat{E}_\phi^2 \\ \hat{B}_r^2 \end{bmatrix} + e^{-i(\omega_2 t+k_{22}z)} \begin{bmatrix} \hat{B}_z^{2*} \\ \hat{E}_\phi^{2*} \\ \hat{B}_r^{2*} \end{bmatrix} \qquad \ldots(1.19)$$

where

$$\left.\begin{aligned} \hat{B}_z^2 &= \frac{1}{2}\,\bar{B}_z^2 J_0(k_{12}r)e^{i\alpha_2} \\ \hat{E}_\phi^2 &= \frac{1}{2i}\,\bar{E}_\phi^2 J_1(k_{12}r)e^{i\alpha_2} \\ \hat{B}_r^2 &= \frac{1}{2i}\,\bar{B}_r^2 J_1(k_{12}r)e^{i\alpha_2} \end{aligned}\right\}(1.20)$$

CHAPTER 2 NONLINEAR INTERACTION OF WAVES IN AN UNMAGNETIZED PLASMA

2.1 Introduction

It is assumed in this chapter that a TE_{0m} and a TE_{0n} wave of the type discussed in Art. 1.3 propagate in an unmagnetized plasma confined to a cylindrical waveguide. Since $\underline{E}$ and $\underline{B}$ are unknown, equations (1.1) and (1.2) must be solved in a self-consistent manner with Maxwell's Equations, hence an extremely complicated set of nonlinear, coupled partial differential equations must be dealt with. Due to the nonlinearities, superposition principles do not hold (the plasma mixes the incoming waves through the terms $(\underline{v}\cdot\nabla)\underline{v}$ and $\underline{j}\times\underline{B}$), hence sum and difference frequency waves are generated. Under certain conditions, significant oscillations are stimulated at the difference frequency, $\omega_1 - \omega_2$. To investigate the nature of these conditions and the size of the oscillations, the equations describing the plasma are linearized by assuming an expansion to second order of the form:

$$\begin{bmatrix} n \\ \underline{v} \\ \underline{E} \\ \underline{B} \end{bmatrix} = \begin{bmatrix} n_o \\ 0 \\ 0 \\ 0 \end{bmatrix} + e^{i(\omega_1 t + k_{21} z)} \begin{bmatrix} n_1 \\ \underline{v}_1 \\ \underline{E}^1 \\ \underline{B}^1 \end{bmatrix} + e^{-i(\omega_1 t + k_{21} z)} \begin{bmatrix} n_1^* \\ \underline{v}_1^* \\ \underline{E}^{1*} \\ \underline{B}^{1*} \end{bmatrix}$$

$$+ e^{i(\omega_2 t + k_{22} z)} \begin{bmatrix} n_2 \\ \underline{v}_2 \\ \underline{E}^2 \\ \underline{B}^2 \end{bmatrix} + e^{-i(\omega_2 t + k_{22} z)} \begin{bmatrix} n_2^* \\ \underline{v}_2^* \\ \underline{E}^{2*} \\ \underline{B}^{2*} \end{bmatrix}$$

$$+ e^{i(\omega t + kz)} \begin{bmatrix} n_3 \\ \underline{v}_3 \\ \underline{E}^3 \\ \underline{B}^3 \end{bmatrix} + e^{-i(\omega t + kz)} \begin{bmatrix} n_3^* \\ \underline{v}_3^* \\ \underline{E}^{3*} \\ \underline{B}^{3*} \end{bmatrix} \qquad \ldots(2.1)$$

where

$$\omega = \omega_1 - \omega_2 \qquad \ldots(2.2)$$

$$k = k_{21} - k_{22} \qquad \ldots(2.3)$$

In equation (2.1), the functional dependence of n, $\underline{v}$, $\underline{E}$, and $\underline{B}$ on r is understood. First order terms are denoted by 1 or 2 - i.e: terms which vary in time at frequency $\omega_1$or ω_2. Second order terms are denoted by 3.

2.2 Collision Terms

The primary interest in this chapter is to stimulate oscillations near the electron-plasma frequency, hence it is assumed that the difference frequency, ω, is sufficiently high that the motion of the ions can be neglected. Equation (1.2) then reduces to an equation of motion for the electrons of the type considered by Salat (3):

$$\frac{\partial \underline{v}_e}{\partial t} + (\underline{v}_e \cdot \nabla)\underline{v}_e = \frac{q_e}{m_e}\left(\underline{E} + \frac{\underline{v}_e \times \underline{B}}{c}\right) - \nu_{ei}\underline{v}_e \qquad \ldots(2.4)$$

where ν_{ei} is the electron-ion collision frequency.

2.3 First Order Fields and Velocities

Since it has been assumed that the incident fields are unaffected by the presence of the plasma, comparing equations (1.17) and (1.19) to equation (2.1) gives:

$$\underline{E}^1 = \underline{u}_\phi E^1_\phi \qquad \ldots(2.5)$$

$$\underline{E}^{2*} = \underline{u}_\phi E^{2*}_\phi \qquad \ldots(2.6)$$

$$\underline{B}^1 = \underline{u}_r B^1_r + \underline{u}_z B^1_z \qquad \ldots(2.7)$$

$$\underline{B}^{2*} = \underline{u}_r B^{2*}_r + \underline{u}_z B^{2*}_z \qquad \ldots(2.8)$$

where $(\underline{u}_r, \underline{u}_\phi, \underline{u}_z)$ form basis vectors for the space defined by (r, ϕ, z).

Substituting equation (2.1) into the equation of motion (2.4) and equating terms of dependence $e^{i(\omega_1 t + k_{21} z)}$ gives:

$$\underline{v}_1 = - \frac{iq}{m_e \omega_1} \underline{E}^1 \qquad \ldots(2.9)$$

Equating terms of dependence $e^{-i(\omega_2 t + k_{22} z)}$ gives:

$$\underline{v}^*_2 = \frac{iq}{m_e \omega_2} \underline{E}^{2*} \qquad \ldots(2.10)$$

2.4 Derivation of Second Order Currents

Second order terms are considered by equating terms of dependence $e^{i(\omega t+kz)}$ when equation (2.1) is substituted into equation (2.4). The result is:

$$i(\omega - i\nu_{ei})\underline{v}_3 + (\underline{v}_1 \cdot \nabla_t)\underline{v}_2^* - i(\underline{k}_2 \cdot \underline{v}_1)\underline{v}_2^* + (\underline{v}_2^* \cdot \nabla_t)\underline{v}_1 + i(\underline{k}_1 \cdot \underline{v}_2^*)\underline{v}_1$$

$$= \frac{q_e}{m_e}\underline{E}^3 + \frac{q_e}{m_e c}\left[\underline{v}_1 \times \underline{B}^{2*} + \underline{v}_2^* \times \underline{B}^1\right] \qquad \ldots(2.11)$$

where

$$\underline{k}_1 = \underline{u}_z k_{21}$$

$$\underline{k}_2 = \underline{u}_z k_{22}$$

∇_t denotes the component of the ∇-operator which is perpendicular to $\underline{u}_z$.

Making use of equations (2.5) to (2.10), equation (2.11) is now considered term by term.

a] $$(\underline{v}_1 \cdot \nabla_t)\underline{v}_2^* = \left[\frac{q_e}{m_e}\right]^2 \frac{(\underline{E}^1 \cdot \nabla_t)\underline{E}^{2*}}{\omega_1 \omega_2} = \left[\frac{q_e}{m_e}\right]^2 \hat{R}\underline{u}_r \qquad \ldots(2.11a)$$

where $$\hat{R} = -\frac{1}{\omega_1 \omega_2}\frac{\hat{E}_\phi^1 \hat{E}_\phi^{2*}}{r} \qquad \ldots(2.11b)$$

b] $$(\underline{k}_2 \cdot \underline{v}_1)\underline{v}_2^* = -i\left[\frac{q_e}{m_e}\right]^2 \frac{k_{22}}{\omega_1 \omega_2}(\underline{u}_z \cdot \underline{E}^1)\underline{E}^{2*} = 0 \qquad \ldots(2.11c)$$

c] $$(\underline{v}_2^* \cdot \nabla_t)\underline{v}_1 = \left[\frac{q_e}{m_e}\right]^2 \frac{1}{\omega_1 \omega_2}(\underline{E}^{2*} \cdot \nabla_t)\underline{E}^1 = \left[\frac{q_e}{m_e}\right]^2 \hat{R}\underline{u}_r \qquad \ldots(2.11d)$$

d] $$(\underline{k}_1 \cdot \underline{v}_2^*)\underline{v}_1 = -i\left[\frac{q_e}{m_e}\right]^2 \frac{k_{21}}{\omega_1\omega_2}(\underline{u}_z \cdot \underline{E}^{2*})\underline{E}^1 = 0 \qquad \text{...(2.11e)}$$

e] $$\frac{q_e}{m_e c}(\underline{v}_1 \times \underline{B}^{2*}) = -i\left[\frac{q_e}{m_e}\right]^2 \frac{1}{\omega_1 c}(\underline{u}_r \hat{E}_\phi^1 \hat{B}_z^{2*} - \underline{u}_z \hat{E}_\phi^1 \hat{B}_r^{2*})$$

$$\frac{q_e}{m_e c}(\underline{v}_2^* \times \underline{B}^1) = i\left[\frac{q_e}{m_e}\right]^2 \frac{1}{\omega_2 c}(\underline{u}_r \hat{E}_\phi^{2*} \hat{B}_z^1 - \underline{u}_z \hat{E}_\phi^{2*} \hat{B}_r^1)$$

Hence

$$\frac{q_e}{m_e c}(\underline{v}_1 \times \underline{B}^{2*} + \underline{v}_2^* \times \underline{B}^1) = \left[\frac{q_e}{m_e}\right]^2 \left[\hat{S}\underline{u}_r + \hat{Q}\underline{u}_z\right] \qquad \text{...(2.11f)}$$

where

$$\hat{S} = -\frac{i}{c}\left[\frac{\hat{E}_\phi^1 \hat{B}_z^{2*}}{\omega_1} - \frac{\hat{E}_\phi^{2*} \hat{B}_z^1}{\omega_2}\right] \qquad \text{...(2.11g)}$$

$$\hat{Q} = \frac{i}{c}\left[\frac{\hat{E}_\phi^1 \hat{B}_r^{2*}}{\omega_1} - \frac{\hat{E}_\phi^{2*} \hat{B}_r^1}{\omega_2}\right] \qquad \text{...(2.11h)}$$

Substituting equations (2.11a) to (2.11h) into equation (2.11) gives:

$$i(\omega - i\nu_{ei})\underline{v}_3 = \frac{q_e \underline{E}^3}{me} + \left[\frac{q_e}{me}\right]^2 \hat{P}\underline{u}_r + \left[\frac{q_e}{me}\right]^2 \hat{Q}\underline{u}_z \qquad \text{...(2.12)}$$

where

$$\hat{P} = \hat{S} - 2\hat{R} \qquad \text{...(2.13)}$$

It can be readily shown that:

$$\hat{P} = -\frac{1}{4}\frac{1}{\omega_1\omega_2}\bar{E}_\phi^1 \bar{E}_\phi^2 e^{i(\alpha_1 - \alpha_2)} \frac{\partial}{\partial r}[J_1(k_{11}r)J_1(k_{12}r)] \qquad \text{...(2.14)}$$

and

$$\hat{Q} = -\frac{i}{4}\frac{1}{\omega_1\omega_2}\bar{E}_\phi^1 \bar{E}_\phi^2 e^{i(\alpha_1 - \alpha_2)} k J_1(k_{12}r)J_1(k_{11}r) \qquad \text{...(2.15)}$$

Note, from equations (2.14) and (2.15):

$$\frac{\partial \hat{Q}}{\partial r} = ik\hat{P} \qquad \ldots(2.16)$$

Reducing equation (2.12) to its radial, angular and axial components gives:

$$\left.\begin{aligned}
i(\omega - i\nu_{ei})v_{3r} &= \frac{q_e}{m_e}E_r^3 + \left[\frac{q_e}{m_e}\right]^2\hat{P} \\
i(\omega - i\nu_{ei})v_{3\phi} &= \frac{q_e}{m_e}E_\phi^3 \\
i(\omega - i\nu_{ei})v_{3z} &= \frac{q_e}{m_e}E_z^3 + \left[\frac{q_e}{m_e}\right]^2\hat{Q}
\end{aligned}\right\} \quad (2.17)$$

or, in terms of currents:

$$\left.\begin{aligned}
j_{3r} &= \frac{a}{4\pi i}E_r^3 + \frac{b}{4\pi i}\bar{P} \\
j_{3\phi} &= \frac{a}{4\pi i}E_\phi^3 \\
j_{3z} &= \frac{a}{4\pi i}E_z^3 + \frac{b}{4\pi i}\bar{Q}
\end{aligned}\right\} \quad (2.18)$$

where

$$\underline{j}_3 = n_o q_e \underline{v}_3 \qquad \ldots(2.19)$$

and

$$\left.\begin{aligned}
a &= \frac{\omega_{pe}^2}{\omega - i\nu_{ei}} \\
b &= -\frac{\omega_{pe}^2}{m_e(\omega - i\nu_{ei})}
\end{aligned}\right\} \quad (2.20)$$

and

$$\left.\begin{aligned}
\bar{P} &= |e|\hat{P} \\
\bar{Q} &= |e|\hat{Q}
\end{aligned}\right\} \quad (2.21)$$

Note that equation (2.19) is valid because n_1 and n_2 are both zero; the plasma density is not perturbed to first order. This follows from the equation of continuity (1.1) for electrons.

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\underline{v}_e) = 0 \qquad \ldots(2.22)$$

Substituting equation (2.1) into equation (2.22) and equating terms of dependence $e^{i(\omega_1 t + k_{21} z)}$ gives:

$$i\omega_1 n_1 + n_o(i\underline{k}_{21}\cdot\underline{v}_1) + n_o(\nabla_t\cdot\underline{v}_1) = 0 \qquad \ldots(2.23)$$

Since $\underline{v}_1$ is in the ϕ-direction only, and is not a function of ϕ or z, then the last two terms of equation (2.23) are zero. Hence $n_1 = 0$. The same is true for n_2.

2.5 Differential Equations for Second Order Fields

From Maxwell's Equations, the following relationships can be derived (see Appendix 1).

$$\left.\begin{aligned}
&\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_r^3)\right] + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]E_r^3 \\
&\qquad = i\frac{4\pi}{\omega}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rj_{3r})\right) + \left(\frac{\omega}{c}\right)^2 j_{3r} + ik\frac{\partial j_{3z}}{\partial r}\right] \\
&\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi^3)\right] + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]E_\phi^3 = i\frac{4\pi\omega}{c^2}\,j_{3\phi} \\
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z^3}{\partial r}\right) + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]E_z^3 \\
&\qquad = i\frac{4\pi}{\omega}\left[-k^2 j_{3z} + \left(\frac{\omega}{c}\right)^2 j_{3z} + ik\,\frac{1}{r}\frac{\partial}{\partial r}(rj_{3r})\right]
\end{aligned}\right\} \quad (2.24)$$

Substituting equations (2.18) into equations (2.24) gives:

$$\left(1-\frac{a}{\omega}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}(rE_r^3)\right) + \left[\left(\frac{\omega}{c}\right)^2\left(1-\frac{a}{\omega}\right) - k^2\right]E_r^3 - ik\frac{a}{\omega}\,\frac{\partial E_z^3}{\partial r}$$
$$= \frac{b}{\omega}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}(r\bar{P})\right) + \left(\frac{\omega}{c}\right)^2\bar{P} + ik\frac{\partial\bar{Q}}{\partial r}\right] \qquad \ldots(2.25)$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}(rE_\phi^3)\right) + \left[\left(\frac{\omega}{c}\right)^2\left(1-\frac{a}{\omega}\right) - k^2\right]E_\phi^3 = 0 \qquad \ldots(2.26)$$

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial E_z^3}{\partial r}\right) + \left(\left(\frac{\omega}{c}\right)^2-k^2\right)\left(1-\frac{a}{\omega}\right)E_z^3 - ik\frac{a}{\omega}\,\frac{1}{r}\,\frac{\partial}{\partial r}(rE_r^3)$$
$$= \frac{b}{\omega}\left[\left(\left(\frac{\omega}{c}\right)^2-k^2\right)\bar{Q} + ik\frac{1}{r}\,\frac{\partial}{\partial r}(r\bar{P})\right] \qquad \ldots(2.27)$$

Consider equation (2.26). There is neither a driving function to excite, nor coupling of the other field components, E_r^3 and E_z^3, to, E_ϕ^3. The only possible solution is the homogenous solution, however, since initially there are no second order fields, then $E_\phi^3 \equiv 0$. Hence, using equation (2.16), the set of equations to be solved reduces to:

$$\left(1-\frac{a}{\omega}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}(rE_r^3)\right) + \left[\left(\frac{\omega}{c}\right)^2\left(1-\frac{a}{\omega}\right) - k^2\right]E_r^3 - ik\frac{a}{\omega}\,\frac{\partial E_z^3}{\partial r}$$
$$= \frac{b}{\omega}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}(r\bar{P})\right) + \left(\left(\frac{\omega}{c}\right)^2-k^2\right)\bar{P}\right] \qquad \ldots(2.28)$$

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial E_z^3}{\partial r}\right) + \left(\left(\frac{\omega}{c}\right)^2-k^2\right)\left(1-\frac{a}{\omega}\right)E_z^3 - ik\frac{a}{\omega}\,\frac{1}{r}\,\frac{\partial}{\partial r}(rE_r^3)$$
$$= \frac{b}{\omega}\left[\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial\bar{Q}}{\partial r}\right) + \left(\frac{\omega}{c}{}^2-k^2\right)\bar{Q}\right] \qquad \ldots(2.29)$$

2.6 Solution of Second Order Field Equations

The approach used to solve for the fields, E_r^3 and E_z^3, involves expanding E_z^3 into a complete set of orthonormal functions which are consistent with the boundary condition, $E_z^3(r=r_o) = 0$. Taking into account the form of equation (2.29), a Fourier-Bessel expansion of zeroeth order Bessel functions appears appropriate. Hence it is assumed that:

$$E_z^3(r) = \sum_{i=1}^{\infty} A_i J_o(r\xi_i) \qquad \ldots(2.30)$$

where

$$J_o(r_o\xi_i) = 0 \qquad \ldots(2.31)$$

or

$$\xi_i = \frac{s_{oi}}{r_o} \qquad \ldots(2.32)$$

where s_{oi} is the i^{th} zero of a zeroeth order Bessel function. The problem remaining is to determine the Fourier-Bessel coefficients A_i.

Substituting equation (2.30) into equation (2.29) gives:

$$ik\frac{a}{\omega}\frac{1}{r}\frac{\partial}{\partial r}(rE_r^3) = \sum_{i=1}^{\infty} A_i\Gamma_i J_o(r\xi_i) - R_2(r) \qquad \ldots(2.33)$$

where

$$\Gamma_i = \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]\left[1 - \frac{a}{\omega}\right] - \xi_i^2 \qquad \ldots(2.34)$$

$$R_2(r) = \frac{b}{\omega}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{Q}}{\partial r}\right) + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]\bar{Q}\right] \qquad \ldots(2.35)$$

Integrating equation (2.33) from 0 to r gives:

$$ik\frac{a}{\omega}E_r^3 = \sum_{i=1}^{\infty}\frac{A_i\Gamma_i}{\xi_i}J_1(r\xi_i) - \frac{1}{r}\int_0^r r'R_2(r')dr' \qquad \ldots(2.36)$$

Note, from equation (2.30):

$$\frac{\partial E_z^3}{\partial r} = -\sum_{i=1}^{\infty}A_i\xi_i J_1(r\xi_i) \qquad \ldots(2.37)$$

Substituting equations (2.36) and (2.37) into equation (2.28) gives:

$$\sum_{i=1}^{\infty}\frac{A_i\Delta_i}{\xi_i}J_1(r\xi_i) = ik\frac{a}{\omega}R_1(r) + \left(1-\frac{a}{\omega}\right)\frac{\partial R_2(r)}{\partial r}$$
$$+\left[\left(\frac{\omega}{c}\right)^2\left(1-\frac{a}{\omega}\right) - k^2\right]\frac{1}{r}\int_0^r r'R_2(r')dr' \qquad \ldots(2.38)$$

where

$$R_1(r) = \frac{b}{\omega}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\bar{P})\right) + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]\bar{P}\right] \qquad \ldots(2.39)$$

and

$$\Delta_i = \Gamma_i\left\{\left[\left(\frac{\omega}{c}\right)^2-\xi_i^2\right]\left[1-\frac{a}{\omega}\right] - k^2\right\} - k^2\xi_i^2\left(\frac{a}{\omega}\right)^2$$
$$= \left[1-\frac{a}{\omega}\right]\left[\left(\frac{\omega}{c}\right)^2\left(1-\frac{a}{\omega}\right) - \xi_i^2 - k^2\right]\left[\left(\frac{\omega}{c}\right)^2 - \xi_i^2 - k^2\right] \qquad \ldots(2.40)$$

Multiplying equation (2.38) by r and differentiating with respect to r gives:

$$\sum_{i=1}^{\infty}A_i\Delta_i J_0(r\xi_i) = S(r) \qquad \ldots(2.41)$$

where

$$S(r) = ik\frac{a}{\omega}\frac{1}{r}\frac{\partial}{\partial r}(rR_1(r)) + \left[\left(\frac{\omega}{c}\right)^2\left(1-\frac{a}{\omega}\right) - k^2\right]R_2(r)$$
$$+\left(1-\frac{a}{\omega}\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R_2(r)}{\partial r}\right) \qquad \ldots(2.42)$$

Using the orthoganality property:

$$\int_0^{r_o} rJ_0(r\xi_i)J_0(r\xi_j)dr = 0 \qquad i \neq j$$

$$= \frac{r_o^2}{2} J_1^2(r_o\xi_j) \qquad i = j$$

It may readily be shown that:

$$A_i = \frac{2}{r_o^2}\frac{1}{\Delta_i}\frac{1}{J_1^2(r_o\xi_i)}\int_0^{r_o} rJ_0(r\xi_i)S(r)dr \qquad \ldots(2.43)$$

Hence, from equation (2.30):

$$E_z^3(r) = \frac{2}{r_o^2}\sum_{i=1}^{\infty}\frac{J_0(r\xi_i)}{\Delta_i J_1^2(r_o\xi_i)}\int_0^{r_o} rJ_0(r\xi_i)S(r)dr \qquad \ldots(2.44)$$

and, from equation (2.36)

$$E_r^3(r) = \frac{\omega}{a}\frac{1}{ik}\left[\frac{2}{r_o^2}\sum_{i=1}^{\infty}\frac{\Gamma_i J_1(r\xi_i)}{\xi_i\Delta_i J_1^2(r_o\xi_i)}\int_0^{r_o} rJ_0(r\xi_i)S(r)dr - \frac{1}{r}\int_0^{r} r'R_2(r')dr'\right] \qquad \ldots(2.45)$$

2.7 Optimization of Second Order Fields

As mentioned earlier, a resonance effect near the electron-plasma frequency has been predicted theoretically for an infinite plasma. It will now be shown how the resonance phenonema can be accounted for in a bounded plasma.

Consider the expression for Δ_i - equation (2.40). Both E_z^3 and E_r^3 are proportional to $1/(1-a/\omega)$. From equation (2.20):

$$\frac{a}{\omega} = \frac{\omega_{pe}^2}{\omega^2+\nu_{ei}^2}\left(1 + i\,\frac{\nu_{ei}}{\omega}\right) \qquad \ldots(2.46)$$

Assuming $\omega >> \nu_{ei}$

$$\frac{a}{\omega} \simeq \left(\frac{\omega_{pe}}{\omega}\right)^2\left(1 + i\frac{\nu_{ei}}{\omega}\right) \qquad \ldots(2.47)$$

If $\omega = \omega_{pe}$, then $1 - \frac{a}{\omega} = - i \frac{\nu_{ei}}{\omega_{pe}}$. Since E_z^3 and E_r^3 are then proportional to ω_{pe}/ν_{ei}, it can be seen that the possibility exists for a significant peak. In general, a maximum will occur when:

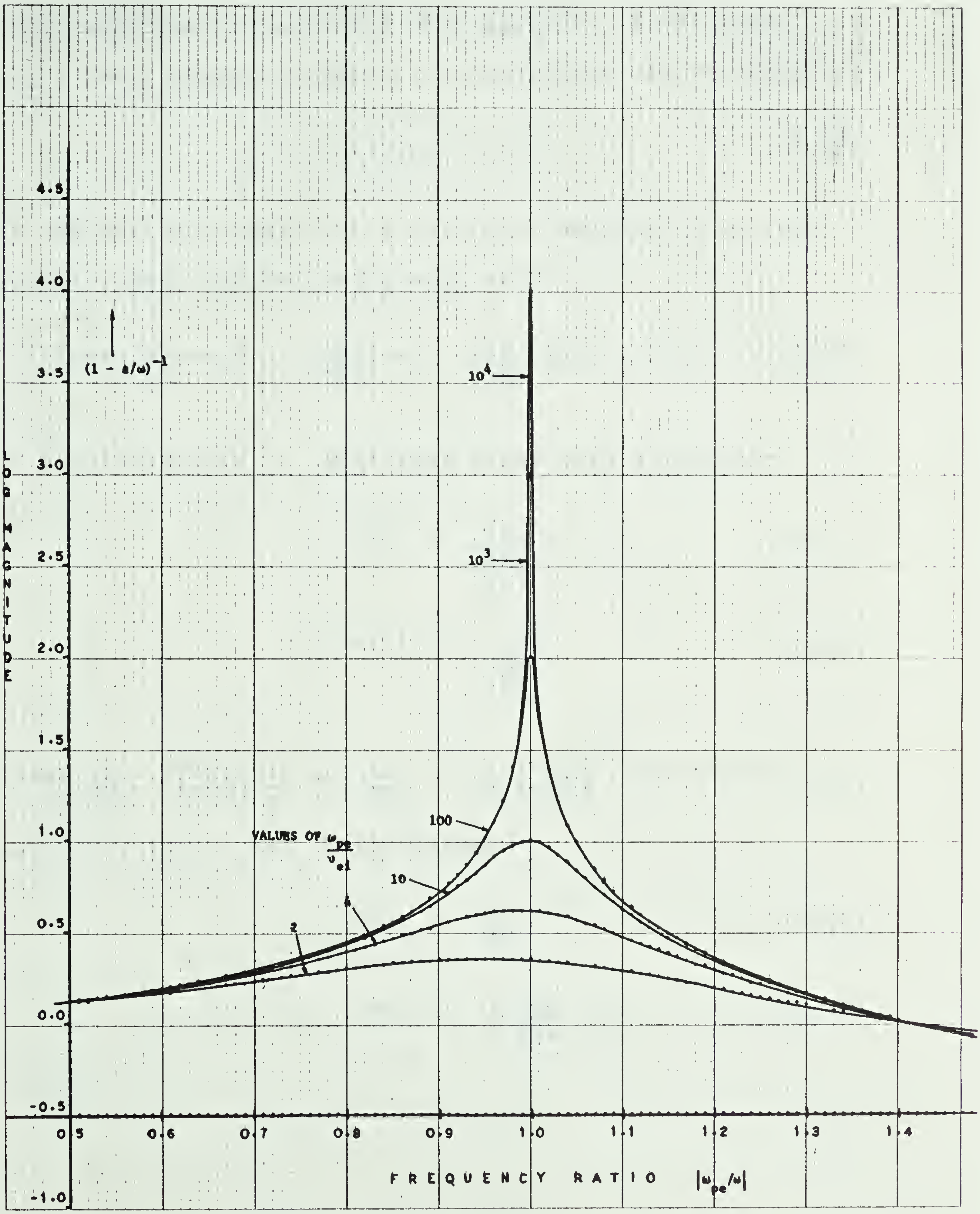
$$|\omega| = \frac{\omega_{pe}}{2}\left[1 + \left(\frac{\nu_{ei}}{\omega_{pe}} + 1\right)^{1/2}\right] \qquad \ldots(2.48)$$

Fig. 2.1 shows a graph of $|1 - a/\omega|^{-1}$ for various values of ω_{pe}/ν_{ei}, in the region $\omega \sim 0(\omega_{pe})$.

2.8 Sensitivity of Field Solutions Near the Optimum Point

Since the factor $1/(1 - a/\omega)$ is the dominant term affecting the magnitude of the second order fields near the resonance point, a measure of its variation with the difference frequency ω, and the plasma density n_o, will give a good indication of the overall sensitivity of the fields to fluctuations in these parameters.

When $\omega_{pe} < 10\nu_{ei}$ the resonance peak is not too sharp and sensitivity will not be a serious problem, however, when $\omega_{pe} > 10\nu_{ei}$, the peak can become very sharp. In general, the optimum condition can be stated as: $|X+iY|$ be a minimum, where:


4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
LOG MAGNITUDE
$(1 - a/\omega)^{-1}$
10^4
10^3
100
10
4
2
VALUES OF $\frac{\omega_{pe}}{\nu_{ei}}$
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
FREQUENCY RATIO $|\omega_{pe}/\omega|$

$$X = \mathrm{Re}\left(1-a/\omega\right), \qquad Y = \mathrm{Im}\left(1-a/\omega\right)$$

This condition is satisfied when equation (2.48) holds. If $\omega_{pe} > 10\nu_{ei}$, then the optimizing condition can be stated as:

$$X(\omega,n_o) = 0 \qquad \text{...(2.49)}$$

If the optimum condition is not to be departed from too greatly, then the following must hold:

$$|X(\omega+\delta\omega,n_o+\delta n_o)| \simeq \left|\frac{\partial X}{\partial \omega}\right| |\delta\omega| + \left|\frac{\partial X}{\partial n_o}\right| |\delta n_o| \leqslant |Y| \qquad \text{...(2.50)}$$

If the inequality is considered along each axis, then

$$|\delta\omega| \leqslant \frac{|Y|}{\left|\frac{\partial X}{\partial \omega}\right|} \qquad \text{...(2.51)}$$

$$|\delta n_o| \leqslant \frac{|Y|}{\left|\frac{\partial X}{\partial n_o}\right|} \qquad \text{...(2.52)}$$

Since $\frac{\partial X}{\partial \omega} \simeq 2\left(\frac{\omega_{pe}}{\omega}\right)^2 \frac{1}{\omega}$, and $\frac{\partial X}{\partial n_o} \simeq -\frac{1}{n_o}\left(\frac{\omega_{pe}}{\omega}\right)^2$, and since $\omega \simeq \omega_{pe}$, equations (2.51) and (2.52) become:

$$|\delta\omega| \leqslant \frac{\nu_{ei}}{2} \qquad \text{...(2.53)}$$

$$|\delta n_o| \leqslant \frac{1}{2}\frac{\nu_{ei}}{\omega_{pe}} n_o \qquad \text{...(2.54)}$$

2.9 Example Problem 1 - Complete Solution of Second Order Fields in an Unmagnetized Plasma

A complete solution of the average value of second order fields as a function of the difference frequency, ω, is now given. The problem was solved with the aid of an IBM 7040 computer which carried the field expansions defined by equations (2.44) and (2.45) to ten terms. The tenth terms were $\sim 0(10^{-5})$ times the first terms of the series and the sign of successive terms was alternating indicating strong convergence of the solutions.

To give a clear indication of the spectrum of the fields, the plasma was defined as follows:

$$\omega_{pe} = 1.5\times 10^{10} \text{ rad/sec}$$

$$\frac{\omega_{pe}}{\nu_{ei}} = 100$$

The waveguide radius was taken as 2 cm. and the first incident field was assumed to be a TE_{01} wave with a fixed frequency, ω_1, of 9×10^{10} rad/sec. The frequency of the second incident field, also a TE_{01} wave, was varied from 13×10^{10} rad/sec to 6.8×10^{10} rad/sec, below which the field would not propagate in the waveguide.

The magnitude of the average value of the second order fields can be expressed in the form:

$$\begin{aligned} \langle E_r^3 \rangle &= A_r(\omega)\,|E_\phi^1 E_\phi^2| \\ \langle E_z^3 \rangle &= A_z(\omega)\,|E_\phi^1 E_\phi^2| \end{aligned} \qquad \ldots(2.55)$$

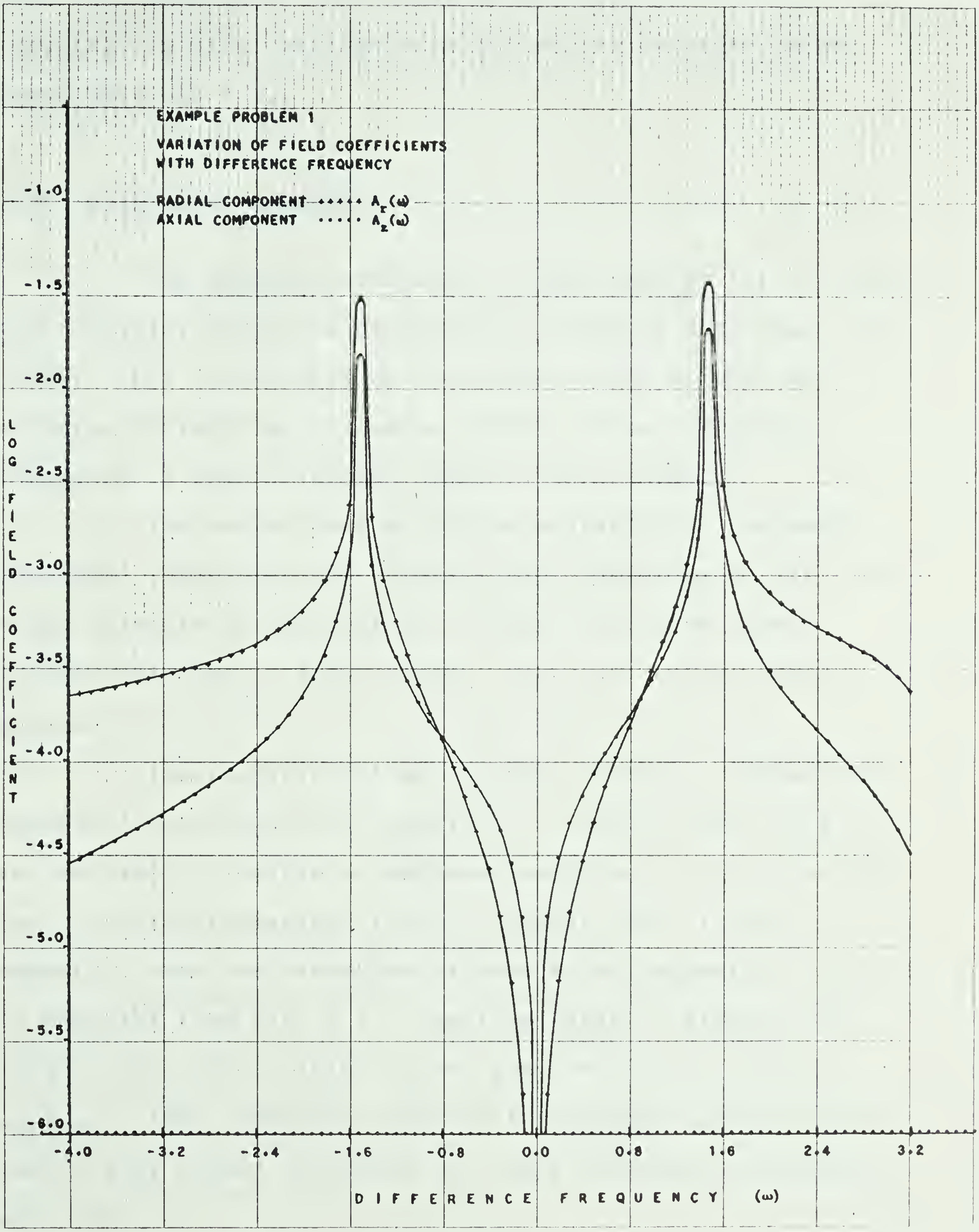

EXAMPLE PROBLEM 1
VARIATION OF FIELD COEFFICIENTS
WITH DIFFERENCE FREQUENCY
RADIAL COMPONENT +++++ $A_r(\omega)$
AXIAL COMPONENT ····· $A_z(\omega)$
LOG FIELD COEFFICIENT
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
-4.0
-4.5
-5.0
-5.5
-6.0
-4.0
-3.2
-2.4
-1.6
-0.8
0.0
0.8
1.6
2.4
3.2
DIFFERENCE FREQUENCY (ω)

Fig. 2.2 illustrates the variation of $\log_{10}[A_r(\omega)]$ and $\log_{10}[A_z(\omega)]$ as a function of ω giving the expected resonances when $|\omega| = \omega_{pe}$.

2.10 Concluding Remarks

The analysis presented in this chapter has verified for a special case, the observations of Stern and Tzoar (5); namely, that significant oscillations can be stimulated in a plasma by two high frequency signals whose difference frequency is near a resonant mode of the plasma.

The use of wave mixing as a diagnostic has been supported theoretically. A particular advantage of the modes selected for excitation is that the plasma density is not perturbed to first order. This will not be true in general.

Equations (2.53) and (2.54) indicate a rather high degree of tuning and low variation in density would have to be achieved to realize an optimum condition, if the electron-ion collision frequency is low. However, this is not a necessity when the procedure is used as a diagnostic. This is apparent from Fig. 2.1. The line shape is almost identical in the lower region of the graph for all values of $\omega_{pe}/\nu_{ei} > 100$, therefore points off resonance can be projected to give a good indication of where the actual resonance will lie.

CHAPTER 3 NONLINEAR INTERACTION OF WAVES IN MAGNETIZED PLASMA

3.1 Introduction

The problem which will now be examined is that of nonlinear interaction of waves in a plasma confined to a cylindrical waveguide with a static magnetic field, B_o, applied axially along the guide. Again, excitation will be by high frequency TE_{0m} and TE_{0n} modes. In contrast to Chapter Two, the difference frequency will be significantly lower than ω_{pe}; sufficiently low that ion motion cannot be neglected. The intent here is to investigate what degree of ion heating may be expected through the wave mixing process.

The magnetic field B_o introduces additional characteristic frequencies defined as cyclotron frequencies.

$$\Omega_\alpha = - \frac{q_\alpha B_o}{m_\alpha c}, \qquad \ldots(3.1)$$

is the cyclotron frequency of the α-species.

The methods used to solve the problem are similar to those employed in Chapter Two, therefore many of the details are not included. The analysis is begun by assuming an expansion similar to equation (2.1), however, both ions and electrons must be accounted for.

$$\begin{bmatrix} n_\alpha \\ \underline{v}_\alpha \\ \underline{E} \\ \underline{B} \end{bmatrix} = \begin{bmatrix} n_{o\alpha} \\ 0 \\ 0 \\ \underline{B}_o \end{bmatrix} + e^{i(\omega_1 t + k_{21} z)} \begin{bmatrix} n_{1\alpha} \\ \underline{v}_{1\alpha} \\ \underline{E}^1 \\ \underline{B}^1 \end{bmatrix} + e^{-i(\omega_1 t + k_{21} z)} \begin{bmatrix} n^*_{1\alpha} \\ \underline{v}^*_{1\alpha} \\ \underline{E}^{1*} \\ \underline{B}^{1*} \end{bmatrix}$$

$$+ e^{i(\omega_2 t + k_{22} z)} \begin{bmatrix} n_{2\alpha} \\ \underline{v}_{2\alpha} \\ \underline{E}^2 \\ \underline{B}^2 \end{bmatrix} + e^{-i(\omega_2 t + k_{22} z)} \begin{bmatrix} n^*_{2\alpha} \\ \underline{v}^*_{2\alpha} \\ \underline{E}^{2*} \\ \underline{B}^{2*} \end{bmatrix}$$

$$+ e^{i(\omega t + k z)} \begin{bmatrix} n_{3\alpha} \\ \underline{v}_{3\alpha} \\ \underline{E}^3 \\ \underline{B}^3 \end{bmatrix} + e^{-i(\omega t + k z)} \begin{bmatrix} n^*_{3\alpha} \\ \underline{v}^*_{3\alpha} \\ \underline{E}^{3*} \\ \underline{B}^{3*} \end{bmatrix} \quad \ldots(3.2)$$

It is desirable that the second order fields be at the cut-off point so that none of the energy available to heat the ions propagates from the plasma. This means that the wavenumber $k = 0$. If this condition is imposed with the additional requirement that the incident waves propagate in the waveguide, it is possible to establish boundaries which restrict the range of ω_1 and ω_2 once the modes of excitation have been selected.

If $k = 0$, then $k_{21} = k_{22}$. From equations (1.10), (1.11), (1.14) and (1.15)

$$r_o^2 = \frac{s_{1m}^2 - s_{1n}^2}{\left(\frac{\omega_1}{c}\right)^2 - \left(\frac{\omega_2}{c}\right)^2} \quad \ldots(3.3)$$

If the incident fields are to propagate then

$$\left(\frac{\omega_1}{c}\right)^2 > \left(\frac{s_{1m}}{r_o}\right)^2, \quad \left(\frac{\omega_2}{c}\right)^2 > \left(\frac{s_{1n}}{r_o}\right)^2$$

Hence

$$1 > \frac{\omega_2}{\omega_1} > \frac{s_{1n}}{s_{1m}} \qquad [s_{1m} > s_{1n}] \qquad \ldots(3.4)$$

$$1 < \frac{\omega_2}{\omega_1} < \frac{s_{1n}}{s_{1m}} \qquad [s_{1m} < s_{1n}] \qquad \ldots(3.5)$$

These conditions are illustrated graphically in Fig. 3.1. The region where all conditions are satisfied is shaded.

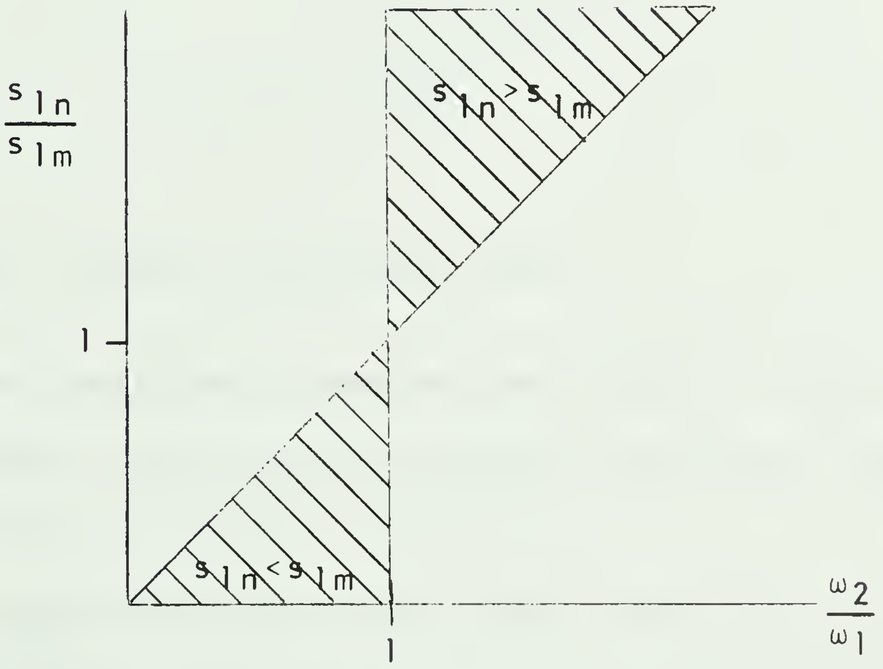


Fig. 3.1 Regions that permit second order cut-off waves and propagating incident waves.

From Fig. 3.1 it can be seen that if identical modes of excitation ($s_{1m} = s_{1n}$) are selected, a trivial solution, $\omega = 0$, results. In general, $\omega_1 \simeq \omega_2$, hence the main region of interest is along the ordinate $\omega_2/\omega_1 = 1$.

[illegible]

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

[illegible]

3.2 Collision Terms

Primary concern in this chapter is the effect of wave mixing on the motion of the ions. The difference frequency ω is sufficiently low that ion motion is significant, hence an equation of motion for both ions and electrons must be considered.

$$\frac{\partial \underline{v}_e}{\partial t} + (\underline{v}_e \cdot \nabla)\underline{v}_e = \frac{q_e}{m_e}\left(\underline{E} + \frac{\underline{v}_e \times \underline{B}}{c}\right) - \nu_{ei}(\underline{v}_e - \underline{v}_i) \qquad \ldots(3.6)$$

$$\frac{\partial \underline{v}_i}{\partial t} + (\underline{v}_i \cdot \nabla)\underline{v}_i = \frac{q_i}{m_i}\left(\underline{E} + \frac{\underline{v}_i \times \underline{B}}{c}\right) + \tilde{\nu}_{ei}(\underline{v}_e - \underline{v}_i) \qquad \ldots(3.7)$$

where

$$\tilde{\nu}_{ei} = m_r \nu_{ei} \qquad \ldots(3.8)$$

3.3 First Order Fields and Velocities

If the additional assumption, $\omega_{1,2} >> \Omega_e$, is made, then the equations for the first order fields are identical to equations (2.5) to (2.8).

The equations for first order velocities are similar to equations (2.9) and (2.10), but both ions and electrons must be taken into account.

$$\underline{v}_{1e} = -\frac{iq_e}{m_e\omega_1}\underline{E}^1, \qquad \underline{v}_{1i} = -\frac{iq_i}{m_i\omega_1}\underline{E}^1$$
$$\underline{v}^*_{2e} = \frac{iq_e}{m_e\omega_2}\underline{E}^{2*}, \qquad \underline{v}^*_{2i} = \frac{iq_i}{m_i\omega_2}\underline{E}^{2*} \qquad \ldots(3.9)$$

3.4 Derivation of Second Order Currents

The equations for the second order currents can be derived in a manner similar to the procedure used to obtain equation (2.12). The magnetic field B_o has no effect on either $\hat{P}$ or $\hat{Q}$ (equations (2.14) and (2.15)), however, since $k = 0$, $\hat{Q} = 0$. The effect of B_o is an additional second order term $\underline{v}_{3e} \times \underline{u}_z B_o$ in the linearized form of equation (3.6) and $\underline{v}_{3i} \times \underline{u}_z B_o$ in equation (3.7). Hence, the second order equation of motion for electrons is

$$i(\omega - i\nu_{ei})\underline{v}_{3e} - \nu_{ei}\underline{v}_{3i} = \frac{q_e}{m_e}\underline{E}^3 + \underline{u}_\phi v_{3er}\Omega_e - \underline{u}_r v_{3e\phi}\Omega_e + \left(\frac{q_e}{m_e}\right)^2 \hat{P}\underline{u}_r \qquad \ldots(3.10)$$

and for ions

$$i(\omega - i\tilde{\nu}_{ei})\underline{v}_{3i} - \tilde{\nu}_{ei}\underline{v}_{3e} = \frac{q_i}{m_i}\underline{E}^3 + \underline{u}_\phi v_{3ir}\Omega_i - \underline{u}_r v_{3i\phi}\Omega_i + \left(\frac{q_i}{m_i}\right)^2 \hat{P}\underline{u}_r \qquad \ldots(3.11)$$

where

$$\frac{q_\alpha(\underline{v}_{3\alpha} \times \underline{u}_z B_o)}{m_\alpha c} = (\underline{u}_r v_{3\alpha\phi} - \underline{u}_\phi v_{3\alpha r})\frac{q_\alpha B_o}{m_\alpha c}$$

$$= \underline{u}_\phi v_{3\alpha r}\Omega_\alpha - \underline{u}_r v_{3\alpha\phi}\Omega_\alpha$$

$\hat{P}$ is defined by equation (2.14).

Equations (3.10) and (3.11) can be reduced in terms of their respective components to yield two matrix equations.

$$\begin{bmatrix} i(\omega - i\nu_{ei}) & -\nu_{ei} \\ -\tilde{\nu}_{ei} & i(\omega - i\tilde{\nu}_{ei}) \end{bmatrix} \begin{bmatrix} v_{3ez} \\ v_{3iz} \end{bmatrix} = \begin{bmatrix} q_e/m_e \\ q_i/m_i \end{bmatrix} E_z^3 \qquad \ldots(3.12)$$

and

$$[\bar{M}][\bar{V}] = [\bar{R}] \qquad \ldots(3.13)$$

where

$$[\bar{M}] = \begin{bmatrix} i(\omega - i\nu_{ei}) & -\nu_{ei} & \Omega_e & 0 \\ -\tilde{\nu}_{ei} & i(\omega - i\tilde{\nu}_{ei}) & 0 & \Omega_i \\ -\Omega_e & 0 & i(\omega - i\nu_{ei}) & -\nu_{ei} \\ 0 & -\Omega_i & \tilde{\nu}_{ei} & i(\omega - i\tilde{\nu}_{ei}) \end{bmatrix} \qquad \ldots(3.14)$$

$$[\bar{V}] = [v_{3er}, v_{3ir}, v_{3e\phi}, v_{3i\phi}]^T \qquad \ldots(3.15)$$

$$[\bar{R}] = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} = \begin{bmatrix} \frac{q_e}{m_e} E_r^3 + \left(\frac{q_e}{m_e}\right)^2 \hat{P} \\ \frac{q_i}{m_i} E_r^3 + \left(\frac{q_i}{m_i}\right)^2 \hat{P} \\ \frac{q_e}{m_e} E_\phi^3 \\ \frac{q_i}{m_i} E_\phi^3 \end{bmatrix} \qquad \ldots(3.16)$$

From equation (3.12), and assuming $\omega^2 >> \nu_{ei}\tilde{\nu}_{ei}$

$$j_{3z} = n_o q_e v_{3ez} + n_o q_i v_{3iz}$$

$$\simeq \frac{\omega_{pe}^2}{4\pi} \frac{E_z^3}{i(\omega - i\nu_{ei})} \qquad \ldots(3.17)$$

Using Mason's Rule, the solution to equation (3.13) can be written as

$$V_\kappa = \frac{1}{|\bar{M}|} \sum_{\iota=1}^{4} (-1)^{\kappa+\iota} R_\iota C_{\kappa\iota} \qquad \ldots(3.18)$$

where V_κ is the κ^{th} element of the matrix $[\bar{V}]$, $|\bar{M}|$, is the determinant of the matrix $[\bar{M}]$, and $C_{\kappa\iota}$ is the cofactor of R_ι in the matrix formed by replacing the κ^{th} column of $[\bar{M}]$ by $[\bar{R}]$. Hence, from equation (3.18), and since $\underline{j}_3 = n_o q_e \underline{v}_{3e} + n_o q_i \underline{v}_{3i}$,

$$j_{3r} = \frac{n_o|e|}{|\bar{M}|} \sum_{\iota=1}^{4} (-1)^{\iota} R_\iota (C_{1\iota} + C_{2\iota}) \qquad \ldots(3.19)$$

$$j_{3\phi} = \frac{n_o|e|}{|\bar{M}|} \sum_{\iota=1}^{4} (-1)^{\iota} R_\iota (C_{3\iota} + C_{4\iota}) \qquad \ldots(3.20)$$

The cofactors in equations (3.19) and (3.20) are now summarized in their exact form.

$$\left.\begin{aligned}
C_{11} &= -i(\omega - i\nu_{ei})[(\omega - i\tilde{\nu}_{ei})^2 - \Omega_i^2] - i\nu_{ei}\tilde{\nu}_{ei}(\omega - i\tilde{\nu}_{ei}) \\
C_{12} &= \nu_{ei}[\Omega_i\Omega_e + (\omega - i\nu_{ei})(\omega - i\tilde{\nu}_{ei})] + \tilde{\nu}_{ei}\nu_{ei}^2 \\
C_{13} &= \Omega_e[(\omega - i\nu_{ei})^2 - \Omega_i^2] - \Omega_i\nu_{ei}\tilde{\nu}_{ei} \\
C_{14} &= i\nu_{ei}(\omega\Omega_i + \omega\Omega_e)
\end{aligned}\right\} \quad (3.21)$$

$$\left.\begin{aligned} C_{21} &= \tilde{\nu}_{ei}[\Omega_i\Omega_e+(\omega-i\nu_{ei})(\omega-i\tilde{\nu}_{ei})] + \nu_{ei}\tilde{\nu}_{ei}^2 \\ C_{22} &= -i(\omega-i\tilde{\nu}_{ei})[(\omega-i\nu_{ei})^2-\Omega_e^2] - i\nu_{ei}\tilde{\nu}_{ei}(\omega-i\nu_{ei}) \\ C_{23} &= i\tilde{\nu}_{ei}(\omega\Omega_i + \omega\Omega_e) \\ C_{24} &= \Omega_i[(\omega-i\nu_{ei})^2-\Omega_e^2] - \Omega_e\nu_{ei}\tilde{\nu}_{ei} \end{aligned}\right\} \quad (3.22)$$

$$\left.\begin{aligned} C_{31} &= -\ C_{13} \\ C_{32} &= -\ C_{14} \\ C_{33} &= \ \ C_{11} \\ C_{34} &= \ \ C_{12} \end{aligned}\right\} (3.23) \qquad \left.\begin{aligned} C_{41} &= -\ C_{23} \\ C_{42} &= -\ C_{24} \\ C_{43} &= \ \ C_{21} \\ C_{44} &= \ \ C_{22} \end{aligned}\right\} (3.24)$$

Also

$$|\bar{M}| = (\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2) - \omega^2\nu^2 + 2i\omega\nu(\Omega_H^2-\omega^2) \qquad \ldots(3.25)$$

where

$$\nu = \nu_{ei} + \tilde{\nu}_{ei} \qquad \ldots(3.26)$$

$$\Omega_H = |\Omega_i\Omega_e|^{1/2} \qquad \ldots(3.27)$$

If values for R_ι are substituted into equations (3.19) and (3.20), the result is

$$\left.\begin{aligned} j_{3r} &= \frac{i}{4\pi}\, a_1 E_r^3 + \frac{1}{4\pi}\, a_2 E_\phi^3 + \frac{i}{4\pi}\, b_1 \bar{P} \\ j_{3\phi} &= \frac{-1}{4\pi}\, a_2 E_r^3 + \frac{i}{4\pi}\, a_1 E_\phi^3 + \frac{1}{4\pi}\, b_2 \bar{P} \end{aligned}\right\} (3.28)$$

where

$$\bar{P} = |e|\hat{P} \qquad \ldots(3.29)$$

$$a_1 = -\frac{i}{|M|}\left[\omega_{pe}^2(C_{11}+C_{21}) + \omega_{pi}^2(C_{12}+C_{22})\right]$$

$$= \frac{\omega_{pe}^2}{|M|}\,\omega[\Omega_H^2-\omega^2+i\omega\nu]\,1^+ \qquad \ldots(3.30)$$

$$a_2 = \frac{1}{|M|}\left[\omega_{pe}^2(C_{13}+C_{23}) + \omega_{pi}^2(C_{14}+C_{24})\right]$$

$$= \frac{\omega_{pe}^2}{|M|}\,\omega^2\Omega_e 1^+1^- \qquad \ldots(3.31)$$

$$b_1 = -\frac{i}{|M|}\left[\frac{\omega_{pe}^2}{m_e}(C_{11}+C_{21}) - \frac{\omega_{pi}^2}{m_i}(C_{12}+C_{22})\right]$$

$$= \frac{1}{m_e}\frac{\omega_{pe}^2}{|M|}\,\omega^2(\omega-i\nu)\,1^+1^- \qquad \ldots(3.32)$$

$$b_2 = -\frac{1}{|M|}\left[\frac{\omega_{pe}^2}{m_e}(C_{13}+C_{23}) - \frac{\omega_{pi}^2}{m_i}(C_{14}+C_{24})\right]$$

$$= -\frac{1}{m_e}\frac{\omega_{pe}^2}{|M|}\,\Omega_e[\omega^2(1+m_r^3) - \Omega_i^2 1^+ - i\omega\tilde{\nu}1^+] \qquad \ldots(3.33)$$

where

$$\left.\begin{aligned} 1^+ &= 1 + m_r \\ 1^- &= 1 - m_r \end{aligned}\right\}(3.34)$$

$$\tilde{\nu} = \tilde{\nu}_{ei}1^+ \qquad \ldots(3.35)$$

If the real parameters, ζ_1, ζ_2, β_1, β_2, ξ_1, ξ_2, η_1 and η_2, are defined by

$$\left.\begin{aligned} a_1 &= \zeta_1 + i\beta_1, \qquad & a_2 &= \zeta_2 + i\beta_2 \\ b_1 &= \xi_1 + i\eta_1, \qquad & b_2 &= \xi_2 + i\eta_2 \end{aligned}\right\}(3.36)$$

and if

$$\left.\begin{aligned} \omega, \Omega_i &\gg \tilde{\nu}_{ei} \\ |\omega \pm \Omega_i| &\gg \tilde{\nu}_{ei} \end{aligned}\right\} \quad (3.37)$$

Then

$$\left.\begin{aligned}
\frac{\zeta_1}{\omega} &\simeq \frac{\hat{\omega}_{pe}^2(\Omega_H^2 - \omega^2)}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)} \\
\frac{\zeta_2}{\omega} &\simeq \frac{\hat{\omega}_{pe}^2 \omega \Omega_e l^-}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)} \\
\frac{\beta_1}{\omega} &\simeq \frac{\hat{\omega}_{pe}^2[(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2) - 2(\Omega_H^2-\omega^2)^2]\omega\nu}{(\omega^2-\Omega_e^2)^2(\omega^2-\Omega_i^2)^2} \\
\frac{\beta_2}{\omega} &\simeq -\frac{2\hat{\omega}_{pe}^2(\Omega_H^2-\omega^2)\omega\Omega_e l^- \omega\nu}{(\omega^2-\Omega_e^2)^2(\omega^2-\Omega_i^2)^2} \\
\frac{\xi_1}{\omega} &\simeq \frac{1}{m_e}\frac{\omega_{pe}^2\omega^2 l^+ l^-}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)} \\
\frac{\xi_2}{\omega} &\simeq \frac{1}{m_e}\frac{\Omega_e}{\omega}\frac{\omega_{pe}^2[\omega^2(1+m_r^3) - \Omega_i^2 l^+]}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)} \\
\frac{\eta_1}{\omega} &\simeq \frac{1}{m_e}\frac{\omega_{pe}^2(\omega^4+\omega^2\Omega_e^2 l^{-2}-\Omega_H^4)l^+ l^- \omega\nu}{(\omega^2-\Omega_e^2)^2(\omega^2-\Omega_i^2)^2} \\
\frac{\eta_2}{\omega} &\simeq -\frac{1}{m_e}\frac{\Omega_e}{\omega}\frac{\omega_{pe}^2[\omega^4((2+m_r)l^-+2m_r^3)-\omega^2\Omega_H^2(1+m_r^2)l^+ +\Omega_i^2\Omega_H^2 l^+]\omega\nu}{(\omega^2-\Omega_e^2)^2(\omega^2-\Omega_i^2)^2}
\end{aligned}\right\} \quad (3.38)$$

where

$$\hat{\omega}_{p\alpha}^2 = \omega_{p\alpha}^2 l^+ \qquad \ldots(3.39)$$

3.5 Differential Equations For Second Order Fields

The differential equations for the second order fields are the same as equations (2.22) with k = 0.

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_r^3)\right) + \left(\frac{\omega}{c}\right)^2 E_r^3 = i\frac{4\pi}{\omega}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rj_{3r})\right)\right] + \left(\frac{\omega}{c}\right)^2 j_{3r} \qquad \ldots(3.40)$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi^3)\right) + \left(\frac{\omega}{c}\right)^2 E_\phi^3 = i\frac{4\pi\omega}{c^2} j_{3\phi} \qquad \ldots(3.41)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z^3}{\partial r}\right) + \left(\frac{\omega}{c}\right)^2 E_z^3 = i\frac{4\pi}{\omega}\left(\frac{\omega}{c}\right)^2 j_{3z} \qquad \ldots(3.42)$$

From equation (3.17) it can be seen that j_{3z} is proportional to E_z^3 only. Hence, the only solution to equation (3.42) is the homogenous solution, $E_z^3 \equiv 0$. By substituting equations (3.28) into equations (3.40) and (3.41), the set of equations to be solved becomes

$$\begin{aligned} &\left(1+\frac{a_1}{\omega}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_r^3)\right) + \left(\frac{\omega}{c}\right)^2\left(1+\frac{a_1}{\omega}\right)E_r^3 \\ &\quad - i\frac{a_2}{\omega}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi^3)\right) + \left(\frac{\omega}{c}\right)^2 E_\phi^3\right] \\ &\quad = -\frac{b_1}{\omega}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\bar{P})\right) + \left(\frac{\omega}{c}\right)^2\bar{P}\right] \end{aligned} \qquad \ldots(3.43)$$

$$\begin{aligned} &\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi^3)\right) + \left(\frac{\omega}{c}\right)^2\left(1+\frac{a_1}{\omega}\right)E_\phi^3 + i\frac{a_2}{\omega}\left(\frac{\omega}{c}\right)^2 E_r^3 \\ &\quad = i\frac{b_2}{\omega}\left(\frac{\omega}{c}\right)^2\bar{P} \end{aligned} \qquad \ldots(3.44)$$

It may be noted, E_ϕ^3 exists as a result of coupling between field components caused by the magnetic field B_o.

3.6 Solution of Second Order Field Equations

The following analysis is similar to Art. 2.6. A Fourier-Bessel expansion for E_ϕ^3 consistent with the boundary condition, $E_\phi^3(r=r_o) = 0$, is used. From equation (3.44), an expansion of first order Bessel functions appears appropriate, hence it is assumed that

$$E_\phi^3(r) = \sum_{\kappa=1}^{\infty} A_\kappa J_1(r\lambda_\kappa) \qquad \ldots(3.45)$$

where

$$J_1(r_o\lambda_\kappa) = 0 \qquad \ldots(3.46)$$

or

$$\lambda_\kappa = \frac{s_{1\kappa}}{r_o} \qquad \ldots(3.47)$$

where $s_{1\kappa}$ is the κ^{th} zero of a first order Bessel function.

Following the methods of Chapter Two, it may readily be shown that

$$A_\kappa = \frac{2}{r_o^2} \frac{1}{\Delta_\kappa} \frac{1}{J_0^2(r_o\lambda_\kappa)} \int_0^{r_o} r J_1(r\lambda_\kappa) S(r)\,dr \qquad \ldots(3.48)$$

where

$$S(r) = i\left(\frac{\omega}{c}\right)^2 \left[\frac{b_2}{\omega}\left(1+\frac{a_1}{\omega}\right)+\frac{a_2}{\omega}\frac{b_1}{\omega}\right] \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rP)\right)+\left(\frac{\omega}{c}\right)^2 P\right] \qquad \ldots(3.49)$$

$$\Delta_\kappa = \left[\left(\frac{\omega}{c}\right)^2-\lambda_\kappa^2\right]\left[\left(\left(\frac{\omega}{c}\right)^2\left(1+\frac{a_1}{\omega}\right)-\lambda_\kappa^2\right)\left(1+\frac{a_1}{\omega}\right)-\left(\frac{\omega}{c}\right)^2\left(\frac{a_2}{\omega}\right)^2\right] \qquad \ldots(3.50)$$

The solution for the second order fields is

$$E_\phi^3 = \frac{2}{r_o^2}\sum_{\kappa=1}^{\infty} \frac{J_1(r\lambda_\kappa)}{\Delta_\kappa J_0^2(r_o\lambda_\kappa)} \int_0^{r_o} r J_1(r\lambda_\kappa) S(r)\, dr \qquad \ldots(3.51)$$

$$E_r^3 = i\,\frac{\omega}{a_2}\left(\frac{c}{\omega}\right)^2 \frac{2}{r_o^2}\sum_{\kappa=1}^{\infty} \frac{\Gamma_\kappa J_1(r\lambda_\kappa)}{\Delta_\kappa J_0^2(r_o\lambda_\kappa)} \int_0^{r_o} r J_1(r\lambda_\kappa) S(r)\, dr$$

$$+ \frac{b_2}{a_2}\,\bar{P}(r) \qquad \ldots(3.52)$$

where

$$\Gamma_\kappa = \left(\frac{\omega}{c}\right)^2\left(1 + \frac{a_1}{\omega}\right) - \lambda_\kappa^2 \qquad \ldots(3.53)$$

Equations (3.51) and (3.52) are now written in a form more adaptable to the following sections.

$$E_\phi^3 = 2B \sum_{\kappa=1}^{\infty} \frac{I_\kappa}{\Delta_\kappa} \frac{J_1(r\lambda_\kappa)}{J_0^2(r_o\lambda_\kappa)} \qquad \ldots(3.54)$$

$$E_r^3 = \frac{b_2}{a_2}\,\bar{P}(r) + 2iB\,\frac{\omega}{a_2}\left(\frac{c}{\omega}\right)^2 \sum_{\kappa=1}^{\infty} \frac{\Gamma_\kappa I_\kappa}{\Delta_\kappa} \frac{J_1(r\lambda_\kappa)}{J_0^2(r_o\lambda_\kappa)}$$

$$\ldots(3.55)$$

where

$$B = -\frac{i}{4r_o}\left(\frac{\omega}{c}\right)^2 \frac{|e|}{\omega_1\omega_2}\left[\frac{b_2}{\omega}\left(1+\frac{a_1}{\omega}\right) + \frac{a_2}{\omega}\frac{b_1}{\omega}\right] \bar{E}_\phi^1 \bar{E}_\phi^2 e^{i(\alpha_1-\alpha_2)} \qquad \ldots(3.56)$$

$$\left.\begin{aligned} I_\kappa &= \frac{1}{r_o^2} I_{\kappa 1} + \left(\frac{\omega}{c}\right)^2 I_{\kappa 2} \\ I_{\kappa 1} &= \int_0^1 u J_1(s_{1\kappa}u)\frac{\partial}{\partial u}\left(\frac{1}{u}\frac{\partial}{\partial u}(u f(ur_o))\right) du \\ I_{\kappa 2} &= \int_0^1 u J_1(s_{1\kappa}u) f(ur_o)\, du \end{aligned}\right\} (3.57)$$

$$f(r) = r_o \frac{\partial}{\partial r}[J_1(k_{11}r)J_1(k_{12}r)] \qquad \ldots(3.58)$$

3.7 Energy Absorbed by the Ions

The rate of energy absorption by the ions, W_i, is given by

$$W_i = \mathrm{Re}\left(\hat{J}^*_{3ir}E^3_r + \hat{J}^*_{3i\phi}E^3_\phi\right) \qquad \ldots(3.59)$$

where $\hat{J}_{3ir}$ and $\hat{J}_{3i\phi}$ are the ion components of the total current with source terms excluded. Equations analogous to equations (3.28) can be written for these components.

$$\left.\begin{aligned} \hat{J}_{3ir} &= \frac{i}{4\pi}\, a_1^+ E^3_r + \frac{1}{4\pi}\, a_2^+ E^3_\phi \\ \hat{J}_{3i\phi} &= -\frac{1}{4\pi}\, a_2^+ E^3_r + \frac{i}{4\pi}\, a_1^+ E^3_\phi \end{aligned}\right\} \qquad (3.60)$$

where

$$\begin{aligned} a_1^+ &= -\frac{i}{|\bar{M}|}\,[\omega^2_{pe}C_{21} + \omega^2_{pi}C_{22}] \\ &= \frac{\omega^2_{pe}}{|\bar{M}|}\,\omega[\Omega^2_H - \omega^2 m_r + i\omega\tilde{\nu}] \qquad \ldots(3.61) \end{aligned}$$

$$\begin{aligned} a_2^+ &= \frac{1}{|\bar{M}|}\,[\omega^2_{pe}C_{23} + \omega^2_{pi}C_{24}] \\ &= \frac{\omega^2_{pe}}{|\bar{M}|}\,\Omega_e[\Omega^2_i - \omega^2 m^2_r + i\omega\tilde{\nu}] \qquad \ldots(3.62) \end{aligned}$$

Combining equations (3.59) and (3.60) gives

$$W_i = -\frac{1}{4\pi}\left(|E_r^3|^2+|E_\phi^3|^2\right)\mathrm{Im}(a_1^+) + \frac{1}{2\pi}\mathrm{Im}(E_r^3 E_\phi^{3*})\,\mathrm{Im}(a_2^+) \qquad \ldots(3.63)$$

Under the assumptions of equation (3.37)

$$\mathrm{Im}(a_1^+) \simeq -\frac{\omega^2\omega_{pi}^2(\omega^2 + \Omega_i^2 - 2\Omega_H^2)\nu}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)^2} \qquad \ldots(3.64)$$

$$\mathrm{Im}(a_2^+) \simeq \frac{\omega\Omega_e\omega_{pi}^2(\omega^2 + \Omega_i^2 - 2\omega^2 m_r)\nu}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)^2} \qquad \ldots(3.65)$$

3.7.1 Optimization of Energy Absorbed by the Ions

The procedure used in this section will be similar to that used to optimize the fields in Chapter Two, however, some simplifying assumptions will be made.

Since both E_ϕ^3 and E_r^3 are proportional to $1/\Delta_\kappa$, then the energy dissipated will be proportional to $(1/\Delta_\kappa)^2$. If the real part of $1/\Delta_\kappa$ can be set to zero for some κ, say $\kappa = \iota$, then $(1/\Delta_\iota)^2$ will be proportional to $1/\nu_{ei}^2$, hence W_i will be proportional to $1/\nu_{ei}$ for the term $\kappa = \iota$. Depending on how small ν_{ei} is, a considerable peaking of the energy spectrum may be expected as $\mathrm{Re}(\Delta_\iota) \to 0$. In contrast to the unmagnetized plasma, the portion of Δ_ι contributing this effect, referred to as P_ι, is a function of ι.

$$P_\iota = \left[\left(\frac{\omega}{c}\right)^2\left(1+\frac{a_1}{\omega}\right) - \lambda_\iota^2\right]\left[1+\frac{a_1}{\omega}\right] - \left(\frac{\omega}{c}\right)^2\left(\frac{a_2}{\omega}\right)^2 \qquad \ldots(3.66)$$

The equivalent term for the unmagnetized plasma, since it was not a function of ι, could be factored as common to all terms in the series expansions for the fields. Thus, under optimum conditions, all terms in the series were optimized. For the case now being considered, only a single term can be optimized. This is not a very serious characteristic since the convergence of the series is rapid and higher order terms have very little contribution.

When an exact expression for W_i is considered, the result does not readily lend itself to analysis, hence it will be assumed that ν_{ei} is sufficiently small that under optimum conditions, all terms $\kappa \neq \iota$, can be considered negligible compared to the term $\kappa = \iota$. Equations (3.54) and (3.55) then reduce to

$$E_\phi^3 \simeq 2B \frac{I_\iota}{\Delta_\iota} \frac{J_1(r\lambda_\iota)}{J_0^2(r_o\lambda_\iota)} \qquad \text{...(3.67)}$$

$$E_r^3 \simeq i2B\left(\frac{\omega}{a_2}\right)\left(\frac{c}{\omega}\right)^2 \frac{\Gamma_\iota I_\iota}{\Delta_\iota} \frac{J_1(r\lambda_\iota)}{J_0^2(r_o\lambda_\iota)} \qquad \text{.. (3.68)}$$

Hence

$$W_i = -\frac{1}{4\pi}\left[\left(\frac{c}{\omega}\right)^4 \left|\frac{\omega}{a_2}\right|^2 |\Gamma_\iota|^2 + 1\right] \mathrm{Im}\left(a_1^+\right) |E_\phi^3|^2$$

$$+ \frac{1}{2\pi} \mathrm{Im}\left\{i\left(\frac{c}{\omega}\right)^2 \left(\frac{\omega}{a_2}\right) \Gamma_\iota\right\} \mathrm{Im}\left(a_2^+\right) |E_\phi^3|^2 \qquad \text{...(3.69)}$$

Consider equation (3.66). Under the assumptions of equations (3.37):

$$P_\iota = \left[\left(\frac{\omega}{c}\right)^2\left(1+\frac{\zeta_1}{\omega}\right) - \lambda_\iota^2\right]\left[1+\frac{\zeta_1}{\omega}\right] - \left(\frac{\omega}{c}\right)^2\left(\frac{\zeta_2}{\omega}\right)^2$$
$$+ i\left\{\frac{\beta_1}{\omega}\left[2\left(\frac{\omega}{c}\right)^2\left(1+\frac{\zeta_1}{\omega}\right) - \lambda_\iota^2\right] - 2\frac{\zeta_2}{\omega}\frac{\beta_2}{\omega}\left(\frac{\omega}{c}\right)^2\right\} \qquad \ldots(3.70)$$

If the optimizing condition, $\mathrm{Re}(P_\iota) = 0$, is applied, the following dispersion relation results.

$$\lambda_\iota^2 = \left(\frac{\omega}{c}\right)^2 \frac{\left(1+\frac{\zeta_1}{\omega}\right)^2 - \left(\frac{\zeta_2}{\omega}\right)^2}{\left(1+\frac{\zeta_1}{\omega}\right)}$$
$$= \left(\frac{\omega}{c}\right)^2 \frac{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2) + \hat{\omega}_{pe}^2[\hat{\omega}_{pe}^2+2(\Omega_H^2-\omega^2)]}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2) + \hat{\omega}_{pe}^2(\Omega_H^2-\omega^2)} \qquad \ldots(3.71)$$

This relation establishes an upper bound on ω. As $[(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2) + \hat{\omega}_{pe}^2(\Omega_H^2-\omega^2)] \to 0$, $\lambda_\iota \to \infty$. But λ_ι must also satisfy equation (3.47). Assuming $\omega_{1,2} >> \omega$, and using equation (3.3):

$$\lambda_\iota^2 = \frac{s_{1\iota}^2}{r_o^2} \simeq \sigma_{\iota mn}\frac{\omega\omega_1}{c^2} \qquad \ldots(3.72)$$

where

$$\sigma_{\iota mn} = \frac{2s_{1\iota}^2}{s_{1m}^2 - s_{1n}^2} \qquad \ldots(3.73)$$

Hence it is being implied that $r_o \to 0$ and $\omega_1 \to \infty$. If this limiting condition is not to be reached, then

$$\omega^2 < \frac{\Omega_H^2(\hat{\omega}_{pi}^2 + \Omega_i^2)}{\hat{\omega}_{pi}^2 + \Omega_H^2} < \Omega_H^2 \qquad \ldots(3.74)$$

Substituting equation (3.71) into equation (3.70) gives

$$P_\iota = -i\left(\frac{\omega}{c}\right)^2 \frac{\hat{\omega}_{pe}^2\omega\nu}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)} \; \frac{(\omega^2+\Omega_e^2)(\omega^2+\Omega_i^2)+(\hat{\omega}_{pe}^2+2\Omega_H^2)(\hat{\omega}_{pe}^2-2\omega^2)}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)+\hat{\omega}_{pe}^2(\Omega_H^2-\omega^2)} \qquad \ldots(3.75)$$

It is also necessary to evaluate the term $\left(\frac{\omega}{c}^2 - \lambda_\iota^2\right)$, since $\Delta_\iota = \left(\frac{\omega}{c}^2 - \lambda_\iota^2\right)P_\iota$.

$$\left(\frac{\omega}{c}^2 - \lambda_\iota^2\right) = -\left(\frac{\omega}{c}\right)^2 \frac{\hat{\omega}_{pe}^2(\hat{\omega}_{pe}^2+\Omega_H^2-\omega^2)}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)+\hat{\omega}_{pe}^2(\Omega_H^2-\omega^2)} \qquad ..(3.76)$$

Γ_ι, defined by equation (3.53), may be approximated by its real part.

$$\Gamma_\iota \simeq \left(\frac{\omega}{c}\right)^2\left(1 + \frac{\zeta 1}{\omega}\right) - \lambda_\iota^2$$

$$= \left(\frac{\omega}{c}\right)^2 \frac{\omega^2\Omega_e^2\hat{\omega}_{pe}^4(1^-)^2}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)[(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)+\hat{\omega}_{pe}^2(\Omega_H^2-\omega^2)]} \qquad \ldots(3.77)$$

Then, $\left(\frac{c}{\omega}\right)^2\frac{\omega}{a_2}\Gamma_\iota$, is approximately given by

$$\left(\frac{c}{\omega}\right)^2\frac{\omega}{\zeta_2}\Gamma_\iota = \frac{\omega\Omega_e\hat{\omega}_{pe}^2 1^-}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)+\hat{\omega}_{pe}^2(\Omega_H^2-\omega^2)} \qquad \ldots(3.78)$$

This result is consistent with equations (3.37).

Substituting equations (3.64) and (3.65) into equation (3.69) gives:

$$W_i = \frac{1}{4\pi}\,\frac{\omega_{pi}^2\,\omega\nu}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)}\left\{\omega(\omega^2+\Omega_i^2-2\Omega_H^2)\left[\left(\frac{c}{\omega}\right)^4\left(\frac{\omega}{\zeta_2}\right)^2\Gamma_\iota^2+1\right]\right.$$

$$\left.+\,2\Omega_e(\omega^2+\Omega_i^2-2\omega^2 m_r)\,\mathrm{Im}\left\{i\left(\frac{c}{\omega}\right)^2\frac{\omega}{\zeta_2}\,\Gamma_\iota\right\}\right\}|E_\phi^3|^2 \qquad \ldots(3.79)$$

and, using equation (3.78), this becomes

$$W_i = \frac{1}{4\pi}\,\frac{\omega_{pi}^2\,\omega^2\nu\left[\hat{\omega}_{pe}^4-2(\omega^2-\Omega_e^2)(\hat{\omega}_{pe}^2+\Omega_H^2-\tfrac{1}{2}(\omega^2+\Omega_i^2))\right]}{\left[(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)+\hat{\omega}_{pe}^2(\Omega_H^2-\omega^2)\right]^2}\,|E_\phi^3|^2 \qquad \ldots(3.80)$$

Consider now, the expression for B, (equation (3.56)). B is proportional to $\frac{b_2}{\omega}\left(1+\frac{a_1}{\omega}\right)+\frac{a_2}{\omega}\frac{b_1}{\omega}$, however, this term can be approximated by its real part.

$$\frac{\xi_2}{\omega}\left(1+\frac{\zeta_1}{\omega}\right)+\frac{\xi_1}{\omega}\frac{\zeta_2}{\omega} = \frac{1}{m_e}\frac{\Omega_e}{\omega}\,\frac{\omega_{pe}^2}{(\omega^2-\Omega^2)(\omega^2-\Omega^2)}\left\{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2-\hat{\omega}_{pi}^2)\right.$$

$$\left.+\,\Omega_H^2(\hat{\omega}_{pi}^2+\Omega_i^2)+\omega^2(\omega^2 m_r^2-\hat{\omega}_{pi}^2-2\Omega_i^2)m_r\right\}$$

$$\simeq \frac{1}{m_e}\,\frac{\omega_{pi}^2}{\omega\Omega_i}\,\frac{\omega^2-\omega_{pi}^2}{\omega^2-\Omega_i^2} \qquad \ldots(3.81)$$

In the last approximation terms $\sim 0(m_r)$ were neglected, however, at this stage of the analysis, the approximation has no serious effect on the final result.

Hence, using equations (3.56), (3.75), (3.76), (3.80) and (3.81), the rate of energy absorption by the ions becomes:

$$W_i \simeq \frac{1}{16\pi s_{1\iota}^2}\left(\frac{ec}{m_e}\right)^2 \frac{1}{\nu}\left(\frac{\Omega_i}{\omega}\right)^4 \left[1+2\left(\frac{\Omega_H}{\omega_{pi}}\right)^2\right]\left(\frac{\omega^2-\omega_{pi}^2}{\omega_{pi}^2}\right)^2$$

$$\times \left[1-\left(\frac{\omega}{\Omega_H}\right)^2\left(1+\left(\frac{\Omega_H}{\omega_{pi}}\right)^2\right)\right]|I_\iota|^2 \frac{|\bar{E}_\phi^1 \bar{E}_\phi^2|^2}{\omega_1^2\omega_2^2}\,\frac{J_1^2(r\lambda_\iota)}{J_0^4(r_o\lambda_\iota)} \quad ..(3.82)$$

To reduce W_i to equation (3.82) from equation (3.80), approximations as high as $\sim 0(4m_r)$ were made on component terms, however, the cummulative effects of all approximations tended to cancel. In addition, none of the approximations were of a critical nature.

3.7.2 Maximization of the Optimum Solution

Equation (3.82) may be considered an optimum solution for the rate of energy absorption by the ions. The optimizing condition prevailing is, $\mathrm{Re}\left(P_\iota\right) = 0$, and this specifies relations between the variables r_o, ω, ω_1, B and n_o (see equations (3.71) and (3.72)). Therefore, to maximize W_i, the parameters in equation (3.82) must be varied in accordance with the dispersion relation and within the bounds of validity of the analysis - i.e.: $\omega_{1,2}$ sufficiently large; high phase velocities; equations (3.37) satisfied.

Since $\omega_{1,2} \rightarrow \infty$ as $\omega \rightarrow \Omega_H$, then $W_i \rightarrow 0$. Now consider $\omega < 15\Omega_i$. Then

$$\lambda_\iota^2 \simeq \left(\frac{\omega}{c}\right)^2\left(\frac{\omega_{pi}}{\Omega_i}\right)^2 \quad ...(3.83)$$

$$r_o^2 \simeq s_{1\iota}^2\left(\frac{c}{\omega}\right)^2\left(\frac{\Omega_i}{\omega_{pi}}\right)^2 \qquad \ldots(3.84)$$

$$|I_\iota|^2 \simeq \frac{1}{r_o^4}\left[I_{\iota 1} + s_{1\iota}^2\left(\frac{\Omega_i}{\omega_{pi}}\right)^2 I_{\iota 2}\right]^2 \qquad \ldots(3.85)$$

Hence

$$W_i \simeq \frac{\omega_{pi}}{16\pi s_{1\iota}^6}\,\frac{\omega_{pi}}{\nu}\left(\frac{e}{m_e c}\right)^2\frac{\omega_{pi}^2}{\omega_1^2\omega_2^2}\left[1 + 2\left(\frac{\Omega_H}{\omega_{pi}}\right)^2\right]^2$$

$$\times\left[I_{\iota 1} + s_{1\iota}^2\left(\frac{\Omega_i}{\omega_{pi}}\right)^2 I_{\iota 2}\right]^2 |\bar{E}_\phi^1\bar{E}_\phi^2|^2\,\frac{J_1^2(r\lambda_\iota)}{J_0^4(r_o\lambda_\iota)} \qquad \ldots(3.86)$$

W_i will be a maximum when Ω_i is a maximum and $\omega_{1,2}$ is a minimum. If shielding of the incident waves by electrons is not to be effective, then the minimum value of $\omega_{1,2}$ is roughly

$$\omega_{1,2}\big|_{min} \simeq 3.16\mathrm{Max}\left(\omega_{pe},\Omega_e\right) \qquad \ldots(3.87)$$

The dispersion relation, equation (3.71), is satisfied when

$$\omega = \sigma_{\iota mn}\omega_1\left(\frac{\Omega_i}{\omega_{pi}}\right)^2 \qquad \ldots(3.88)$$

Two cases will now be considered. In the first case it will be assumed that $\omega_{pe} > \Omega_e$, and in the second it will be assumed that $\omega_{pe} < \Omega_e$.

1] $\omega_{pe} > \Omega_e$

Under this assumption

$$\begin{aligned} \Omega_i \big|_{max} &= \sqrt{m_r}\,\omega_{pi} \\ \omega_1 \big|_{min} &= 3.16\omega_{pe} \qquad (3.89) \\ \omega &= 3.16\sigma_{\ell mn}\Omega_i \end{aligned}$$

If this is to be a valid maximum, then there is an upper bound on $\sigma_{\ell mn}$ approximately given by

$$\sigma_{\ell mn} < 4.75 \qquad \ldots(3.90)$$

Otherwise, equation (3.86) does not hold since $\omega > 15\Omega_i$. Assuming equation (3.90) is satisfied, then

$$W_i \simeq \frac{3\times 10^{-2}}{16\pi s_{1\ell}^6}\,\frac{e}{m_i c}\,\frac{1}{\nu}\left[I_{\ell 1} + m_r s_{1\ell}^2 I_{\ell 2}\right]^2 \left|\bar{E}_\phi^1 \bar{E}_\phi^2\right|^2 \frac{J_1^2(r\lambda_\ell)}{J_0^4(r_o\lambda_\ell)} \qquad \ldots(3.91)$$

It must now be determined which term in the exact expression for the energy absorption to optimize (ie: ℓ), and which modes of excitation should be used (ie: m and n). It is desirable to make $F_{\ell mn}$ as large as is practicably possible, where

$$F_{\ell mn} = \frac{1}{s_{1\ell}^6}\left[I_{\ell 1} + m_r s_{1\ell}^2 I_{\ell 2}\right]^2 \frac{J_1^2(r\lambda_\ell)}{J_0^4(r_o\lambda_\ell)} \qquad \ldots(3.92)$$

In Appendix 2, it is shown that for a given m and n, $F_{\iota mn}$ must eventually begin to decrease and approaches zero as ι approaches infinity. The convergence is shown to be rapid and the best choice for ι is $\iota = 1$. If $\iota = 1$, equation (3.90) is satisfied for all m and n, $m \neq n$.

The best selection for m and n is governed by practical limitations. The tables in Appendix 2 indicate that as m and n increase for a fixed value of ι, $F_{\iota mn}$ increases, however, it also becomes more difficult to excite the modes. In addition, the radius of the waveguide becomes large as m and n increase, since when W_i is a maximum:

$$r_o = \frac{1}{2}\left[\frac{m_i}{m_e}\right]^{\frac{1}{2}} \frac{\left|s_{1m}^2 - s_{1n}^2\right|}{s_{1\iota}} \frac{c}{\omega_1} \qquad \ldots(3.93)$$

Figure 3.2 illustrates the variation of $F_{\iota mn}$ for various value of ι, m and n. A trend is evident and this is indicated by continuous lines although the actual points are discrete. Beyond the range considered, the results are practically of little interest, but the points appear to be approaching a limiting value as they must since there is only a finite amount of energy in the incident waves.

2] $\omega_{pe} < \Omega_e$

It will be shown that the results obtainable for this situation will not be as favorable as the case where $\omega_{pe} > \Omega_e$. If $\omega_{pe} < \Omega_e$, then $\omega_{1,2}$ must increase as the

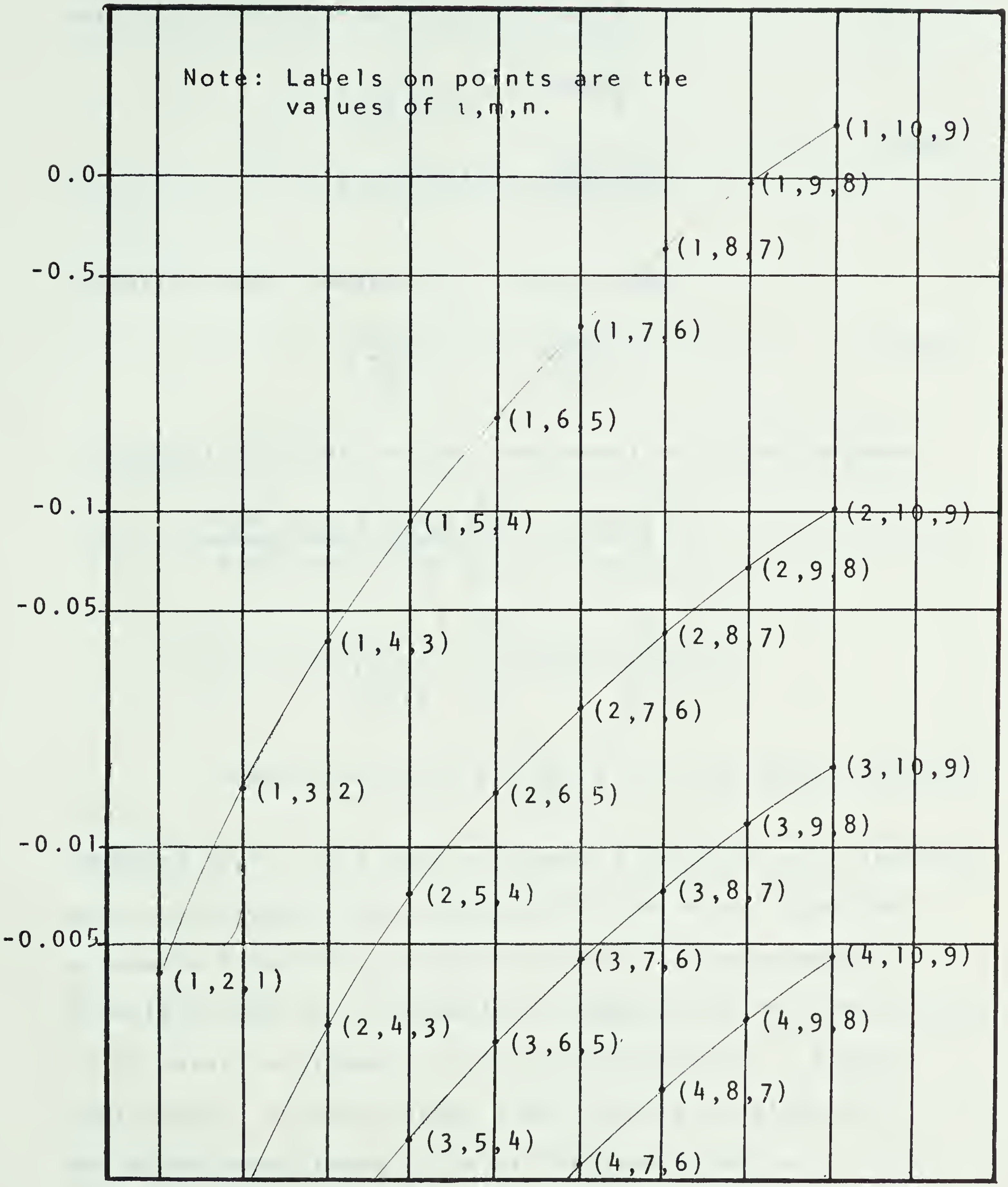


Figure 3.2 Variation of $F_{\iota mn}$

magnetic field intensity is increased.

$$\omega_{1,2}\Big|_{min} = 3.16\Omega_e$$
$$\omega = 3.16\sigma_{lmn}\left(\frac{\Omega_i}{\omega_{pi}}\right)^2\Omega_e \qquad (3.94)$$

To satisfy the inequality $\omega < 15\Omega_i$, then

$$\left(\frac{\Omega_i}{\omega_{pi}}\right)^2 < \frac{4.75}{\sigma_{lmn}} m_r \qquad \ldots(3.95)$$

If inequality (3.95) holds, then equation (3.86) becomes

$$W_i \simeq \frac{1\times10^{-2}}{16\pi s_{1l}^6}\left(\frac{e}{m_i c}\right)^2\frac{1}{\nu}\left(\frac{\omega_{pi}}{\Omega_H}\right)^4\left[1 + 2\left(\frac{\Omega_H}{\omega_{pi}}\right)^2\right]$$
$$\times\left[I_{l1} + s_{1l}^2\left(\frac{\Omega_i}{\omega_{pi}}\right)^2 I_{l2}\right]^2 \left|\bar{E}_\phi^1\bar{E}_\phi^2\right|^2 \frac{J_1^2(r\lambda_l)}{J_0^4(r_o\lambda_l)}$$

Unless the term, $s_{1l}^2\left(\frac{\Omega_i}{\omega_{pi}}\right)^2 I_{l2}$, can become dominant compared to I_{l1}, the rate of energy absorption will decrease as Ω_i increases. The largest value the former term can assume as dictated by equation (3.95) is approximately $2.4m_r(s_{1m}^2-s_{1n}^2)I_{l2}$. The tables in Appendix 2 show that $|I_{l1}|$ is at least ten times $|I_{l2}|$ for all values of l, m and n considered. A conservative limit which will allow the above mentioned terms to be of the same order is

$$|s_{1m}^2 - s_{1n}^2| > 7000$$

To achieve this value, either the TE_{0m} or the TE_{0n} mode

would have to be of order greater than 25. In addition, the radius of the waveguide would be excessive, therefore this case is not a practical consideration.

Hence, case [1] is preferable to case [2] and the fact that a lower magnetic field is used is an added practical advantage.

3.7.3 Sensitivity of W_i Near the Optimum Point

In this section the sensitivity of the optimum solution to small fluctuations in ω, n_o and B_o will be investigated. As in Art. 2.8, this will be done by considering the term from which the optimizing criteria was derived; in this case P_l. Defining the real quantities X_l and Y_l by

$$P_l = X_l + iY_l \qquad \ldots(3.97)$$

then an optimum condition prevails provided X_l does not become large compared to Y_l. The following inequalities should be satisfied

1] $$|\delta\omega| < \frac{|Y_l|}{\left|\frac{\partial X_l}{\partial \omega}\right|} \qquad \ldots(3.98)$$

2] $$|\delta n_o| < \frac{|Y_l|}{\left|\frac{\partial X_l}{\partial n_o}\right|} \qquad \ldots(3.99)$$

3]

$$|\delta B_o| < \frac{|Y_\ell|}{\left|\frac{\partial X_\ell}{\partial B_o}\right|} \qquad \ldots(3.100)$$

To simplify the analysis it will be assumed that equation (3.90) holds so that the assumption, $\omega < 15\Omega_i$, is valid. Then, from equation (3.75)

$$Y_\ell \simeq \left(\frac{\omega}{c}\right)^2\left(\frac{\omega_{pe}}{\Omega_e}\right)^2\left(\frac{\omega_{pe}}{\Omega_H}\right)^2\frac{\omega\nu_{ei}}{\omega^2-\Omega_i^2} \qquad \ldots(3.101)$$

and, from equation (3.70)

$$X_\ell = \left(\frac{\omega}{c}\right)^2\left[1 + \frac{\hat{\omega}_{pe}^2[\hat{\omega}_{pe}^2+2(\Omega_H^2-\omega^2)]}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)}\right] - \lambda_\ell^2\left[1 + \frac{\hat{\omega}_{pe}^2(\Omega_H^2-\omega^2)}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2)}\right]$$

$$\simeq -\frac{1}{\omega^2-\Omega_i^2}\left(\frac{\omega_{pe}}{\Omega_e}\right)^2\left[\left(\frac{\omega}{c}\right)^2\omega_{pe}^2 - \lambda_\ell^2\Omega_H^2\right]$$

$$= -\frac{1}{\omega^2-\Omega_i^2}\left(\frac{\omega_{pe}}{\Omega_e}\right)^2\left[\left(\frac{\omega}{c}\right)^2\omega_{pe}^2 - \sigma_{\ell mn}\frac{\omega\omega_1}{c^2}\,\Omega_H^2\right] \qquad \ldots(3.102)$$

1]

$$\frac{\partial X_\ell}{\partial\omega} \simeq -\frac{1}{\omega}\left(\frac{\omega}{c}\right)^2\left(\frac{\omega_{pe}}{\Omega_e}\right)^2\frac{\omega_{pe}^2}{\omega^2-\Omega_i^2} \qquad \ldots(3.103)$$

Hence variations in ω are restricted within

$$|\delta\omega| < \left(\frac{\omega}{\Omega_H}\right)^2\nu_{ei} \qquad \ldots(3.104)$$

Defining a quality factor, Q, by

$$Q = \frac{\omega_1}{2\delta\omega},$$

then

$$Q \simeq \frac{1}{2}\left(\frac{\Omega_i}{\omega}\right)^2 \frac{\omega_1}{\tilde{\nu}_{ei}} \qquad \ldots(3.105)$$

2] Since n_o is proportional to ω_{pe}^2, the allowable variation in ω_{pe}^2 will be found first.

$$\frac{\partial X_\iota}{\partial \omega_{pe}^2} \simeq - \frac{1}{\omega^2 - \Omega_i^2}\left(\frac{\omega}{c}\right)^2\left(\frac{\omega_{pe}}{\Omega_e}\right)^2 \qquad \ldots(3.106)$$

Hence variations in ω_{pe}^2 are restricted within

$$|\delta\omega_{pe}^2| < \frac{|\omega|\nu_{ei}}{\Omega_H^2}\,\omega_{pe}^2 \qquad \ldots(3.107)$$

Therefore variations in n_o are restricted within

$$|\delta n_o| < \frac{|\omega|\nu_{ei}}{\Omega_H^2}\,n_o \qquad \ldots(3.108)$$

3] Since B_o is proportional to Ω_i, the allowable variation in Ω_i will be found first.

$$\frac{\partial X_\iota}{\partial \Omega_i} \simeq \frac{2}{\Omega_i}\,\frac{\omega_{pe}^2}{\omega^2 - \Omega_i^2}\left(\frac{\omega}{c}\right)^2\left(\frac{\omega_{pe}}{\Omega_e}\right)^2 \qquad \ldots(3.109)$$

Hence variations in Ω_i are restricted within

$$|\delta\Omega_i| < \frac{1}{2}\,\frac{|\omega\Omega_i|}{\Omega_H^2}\,\nu_{ei} \qquad \ldots(3.110)$$

Therefore variations in B_o are restricted within

$$|\delta B_o| < \frac{1}{2}\,\frac{|\omega|\nu_{ei}}{\Omega_H^2}\,|B_o| \qquad \ldots(3.111)$$

3.7.4 Energy Dissipation in a Hot Diffuse Plasma

In order to give a clearer indication of some of the actual numbers involved in ion heating, a hot diffuse plasma will now be considered. It will be assumed that the incident waves are TE_{01} and TE_{02} modes. The plasma is characterized by the following parameters.

$$n_o = 10^{12}\ \text{cm}^{-3}$$
$$T = 10^6\ {}^\circ K$$
$$\omega_{pe} = 5.6\times10^{10}\ \text{sec}^{-1}$$
$$\omega_{pi} = 1.3\times10^{9}\ \text{sec}^{-1}$$
$$v_{\theta e} = 5.5\times10^{8}\ \text{cm-sec}^{-1}$$
$$v_{\theta i} = 1.3\times10^{7}\ \text{cm-sec}^{-1}$$
$$\nu_{ei} = 5.1\times10^{5}\ \text{sec}^{-1}$$

The first term in the series expansion for W_i will be optimized, hence $\iota = 1$. It will also be assumed that $m = 2$, and $n = 1$. Therefore

$$s_{1\iota} = s_{11} \simeq 3.83$$

$$F_{\iota mn} = F_{121} \simeq 4.74\times10^{-3}$$

$$\sigma_{\iota mn} = \sigma_{121} \simeq 0.850$$

Using the formulas developed in Art. 3.7.2 for the case where $\omega_{pe} > \Omega_e$, then

$$\omega_{1,2} \simeq 3.16\omega_{pe} \simeq 1.8\times10^{11}\ \text{sec}^{-1}$$

$$W_i \approx 0.107F_{121}|E_\phi^1 E_\phi^2|^2 \approx 5.1\times10^{-4}|E_\phi^1 E_\phi^2|^2$$

$$r_o \approx \frac{7.24s_{11}}{\sigma_{121}} \approx 32.5 \text{ cm}$$

$$B_o \approx 3.19 \text{ kilogauss}$$

$$\Omega_i \approx 3.02\times10^7 \text{ sec}^{-1}$$

$$\omega \approx 9.54\times10^7 \,\sigma_{121} \approx 8.11\times10^7 \text{ sec}^{-1}$$

From Art. 3.7.3, the sensitivities are

$$Q \approx \frac{3.3\times10^7}{\sigma_{121}^2} \approx 4.6\times10^7$$

$$\left|\frac{\delta B_o}{B_o}\right| \approx 1.44\times10^{-5}\,\sigma_{121} \approx 1.22\times10^{-5}$$

$$\left|\frac{\delta n_o}{n_o}\right| \approx 2.88\times10^{-5}\,\sigma_{121} \approx 2.44\times10^{-5}$$

An instructive way to consider the problem is to determine the approximate time, τ, the sources must be turned on to achieve a 1% increase in temperature. Equating the change in energy of the ions to the energy absorbed by them in time τ

$$n_o m_i v_{\theta i}(\Delta v_{\theta i}) = W_i\tau \qquad \ldots(3.112)$$

The power output of each source is given by

$$U_\ell^s = \frac{c}{8\pi}\int_A \mathrm{Re}\left(\underline{\hat{E}}^\ell\times\underline{\hat{B}}^{\ell *}\right)\cdot d\underline{A} \qquad \ldots(3.113)$$

The integration is carried out over the cross-section of the waveguide and ℓ denotes either source 1 or source 2. From the equations of Art. 1.3

$$U_\ell^s \simeq \frac{cr_o^2}{32} J_0^2(s_{1\ell}) |E_\phi^\ell|^2 \qquad \ldots(3.114)$$

Hence, for the case now being considered

$$\tau \simeq \left(\frac{cr_o^2}{32}\right)^2 \frac{n_o m_i v_{\theta i}(\Delta v_{\theta i})}{5.1\times10^{-4}} \frac{J_0(s_{11})J_0(s_{12})}{U_1^s U_2^s} \qquad \ldots(3.115)$$

A 1% increase in ion temperature requires a 0.5% increase in the thermal velocity of the ions, therefore

$$\Delta v_{\theta i} \simeq 6.5\times10^4 \text{ cm-sec}^{-1}$$

Then, assuming the sources have equal power outputs

$$\tau \simeq \frac{3.25\times10^{26}}{(U^s)^2}$$

To achieve a 1% increase in temperature in 1 second under ideal conditions, sources with an output power of about 1800 kilowatts would be required. Since the output of these sources would have to be pulsed the time will be correspondingly increased. It will be further increased if the required value of Q cannot be realized. Hence, the problem as treated here, under the assumptions made, yields a low rate of heating, however it should be possible to improve these results by several orders of magnitude and

suggestions to this effect follow in Art. 3.8.

As a final point of interest, it will now be checked that the high phase velocity approximation is valid. Since k = 0, the phase velocity in the z-direction is infinity hence the condition (equation (1.4)) is certainly satisfied in this direction. Now consider the direction perpendicular to z.

$$\lambda_\iota \simeq \left|\left(\frac{\omega}{c}\right)\left(\frac{\omega_{pi}}{\Omega_i}\right)\right| \simeq 0.116$$

Hence

$$\frac{|\omega - \Omega_i|}{\lambda_\iota} \simeq 4.35\times10^9 \quad >> \quad v_{\theta i} \simeq 1.3\times10^7$$

$$\frac{|\Omega_e|}{\lambda_\iota} \simeq 5.2\times10^{12} \quad >> \quad v_{\theta e} \simeq 5.5\times10^8$$

Therefore, the high phase velocity approximation is justified for this case.

3.8 Concluding Remarks

Although the degree of heating that can be achieved is small for the problem as considered here, this should not be accepted as indicative of the overall potential. A principal factor contributing to the low value of energy absorption is the large radius that is required to allow a cut-off signal (k = 0), which leads to the consequent high power output of the sources that is necessary to achieve appreciable heating. If this condition is relaxed, then

the radius can be set arbitrarily. Although the available energy is not being used as efficiently as possible, the amount of heating that is realizable will be greater since a smaller confining structure can be used. Thus for sources of a given power output, the energy density in the structure will be greater. A possible method to decrease the loss in efficiency for cases where $k \neq 0$ would be to employ a toroidal or figure eight (stellarator) structure. A further advantage with smaller radii is, the area across which the magnetic field must be applied is less.

The reason for selecting cut-off signals was based on the results of an analysis carried out (6) for an infinite plasma excited by plane waves. The wavevector of the second order fields was defined by a component perpendicular to the direction of the static magnetic field, $k_{\perp}$, and a parallel component, $k_{\parallel}$. It was found that setting $k_{\parallel}$ to zero gave the better results. It is interesting to compare this problem to the problem considered here. The ratio of the rate of energy absorption in the infinite plasma, W_i^{∞}, to the value of W_i given by equation (3.86) is: $W_i^{\infty}/W_i = 4/F_{lmn}$. In general, this ratio is large, as can be seen from the values of F_{lmn} given in Appendix 2. This difference can be related to the equations for the second order fields and arises because

$$\left\langle \frac{\int_0^{r_o} rJ_1(r\lambda_l)\left[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r\bar{P}) + \left(\frac{\omega}{c}\right)^2\bar{P}\right]dr}{\int_0^{r_o} rJ_1^2(r\lambda_l)dr} \times J_1(r\lambda_l)\right\rangle$$

$$<< \left[\left(\frac{\omega}{c}\right)^2 - \bar{P}^\infty\right]$$

where (6)

$$\bar{P}^\infty = -i\frac{|e|k_\perp}{\omega_1\omega_2}(\underline{E}^1\cdot\underline{E}^{2*})$$

In an infinite plasma excited by plane waves, the degree of coupling between the incident waves is constant throughout and $\bar{P}^\infty$ has no spatial dependence. However, in a finite plasma of the type considered here, $\bar{P}$ does have a radial dependence which adversely affects the net amount of coupling. A possible explanation is as follows:

Consider each incident wave as imparting a component of electromagnetic momentum to the second order wave. For the momentum transfer to be a maximum, these components must add over all space; the resultant second order field being proportional to the spatial average of all momentum contributions. In the infinite plasma, these components are fixed and give the same contribution over all space, however, in the finite plasma, these components vary in magnitude and direction and the net effect upon averaging is small.

The underlying reason for the difference in the degree of coupling between the finite and infinite case is in the nature of the waves which excite the plasma. In the infinite plasma we are concerned with the coupling of travelling waves in the direction perpendicular to the magnetic field, whereas in the finite plasma, we are coupling standing waves. The existence of standing waves specifies a discrete (since the modes of excitation can be adjusted discretely) phase relation between the incident waves. To achieve travelling wave coupling in a finite structure, the magnetic field would have to be applied across, rather than along, the structure. Then the incident waves across the magnetic field appear as travelling waves as in the infinite plasma. This particular point is one which deserves detailed investigation. A further area open to study is consideration of other types of geometry for the confined region.

The sensitivity of W_i to small fluctuations in any of the parameters is, at present, somewhat restrictive. To control the density of the plasma to one part in 10^5 is very difficult and is a further justification for the use of smaller structures. The quality factor, Q, is also high. The hot diffuse plasma considered in Art. 3.7.4 required a Q of 4.6×10^7, however, a more desirable figure would be about 10^5 since this is more within the realm of present technology.

It was mentioned in Art. 3.7.4 that, since the power levels involved to achieve significant heating were high, the sources would have to be pulsed rather than having a continuous output. This disadvantage could be slightly lessened by using several, rather than just two, sources.

Assuming a method similar to the type being considered was to be used for ion heating, it should be noted that the rate of heating increases as temperature increases since the collision frequency, ν_{ei}, decreases (or at worst the rate remains constant, since sensitivity increases in direct proportion to $1/\nu_{ei}$). This is in contrast to the method of ion-cyclotron heating discussed by Berger et al (9), where at high temperatures, the rate of heating decreased as temperature increased.

CHAPTER 4 DISCUSSION

The analysis presented in this thesis indicates a suitable approach to some aspects of the problem of nonlinear interaction of high frequency waves in bounded plasmas, once basic assumptions as to the nature of the plasma and what is to be done with it are made. In theory, the problem could be generalized to a greater extent by assuming an arbitrarily shaped boundary and expanding the fields in some set of orthoganal functions, $\Phi(\psi_K,\underline{r})$. (In this thesis $\Phi(\psi_K,\underline{r}) \to J_n(\lambda_K r)$). The main problem in practice, of course, would be to find a tabulated set of orthoganal functions.

One aspect of the problem which requires research, and to the author's knowledge has not yet been investigated is wave coupling in a bounded, non-uniform plasma. This is a more realistic model and it is well known that such a plasma will in general support more than one normal mode of oscillation (10), (11), whereas the uniform plasma has only one normal mode at the plasma frequency. Another factor which contributes to additional resonances is the thermal velocities of the electrons and ions which must be taken into account if phase velocities are small. A further complicating factor which now arises due to these additional considerations is that of correlating the field spectrum to the density distribution if wave mixing is to be used as a

diagnostic. This is a point where the mixing of laser beams enjoys an advantage over the general mixing of waves throughout the plasma. With lasers, the interaction can be investigated in very local regions across which density distribution can be ignored.

With respect to the section on ion heating, although the rate of energy absorption by the ions was small for the case considered, the investigation has indicated the areas to which future studies should be directed. A major limiting factor for the amount of heating that can be achieved is the necessarily high frequency of the incident waves to permit them to thoroughly penetrate the plasma. The amount of heating varies inversely as $\omega_1^2\omega_2^2$. Not only would the heating increase substantially if lower frequencies could be used, but a considerable reduction in Q may be expected. In addition, the method can then be applied to higher density plasmas with direct application to thermonuclear heating. The problem of using lower frequency waves is currently being investigated by C. E. Capjack (12).

In conclusion, there are yet many aspects of the problem of nonlinear coupling of waves in plasmas that require detailed study. Two major applications have been mentioned and there are perhaps many more. In particular, the achieving of significant amounts of ion heating has great potential.

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APPENDIX 1 DERIVATION OF DIFFERENTIAL EQUATIONS FOR SECOND ORDER FIELDS

In this Appendix, differential equations for the the second order fields of the unmagnetized plasma (Chapter 2) are derived. The results are directly applicable to the magnetized plasma (Chapter 3).

From equations (1.5) and (1.8)

$$\nabla \cdot \underline{j}_3 = - i \omega \rho_3 \qquad \text{...(A1.1)}$$

and, from equations (1.7) and (1.8)

$$\nabla(\nabla \cdot \underline{E}^3) - \nabla^2 \underline{E}^3 = -i \frac{4\pi\omega}{c^2} \underline{j}_3 + \left(\frac{\omega}{c}\right)^2 \underline{E}^3 \qquad \text{...(A1.2)}$$

Substituting equation (A1.1) into equation (1.5) gives

$$\nabla \cdot \underline{E}^3 = i \frac{4\pi}{\omega} \left[\frac{1}{r} \frac{\partial}{\partial r}(r j_{3r}) + ik j_{3z} \right] \qquad \text{...(A1.3)}$$

Hence

$$\left.\begin{aligned} [\nabla(\nabla \cdot \underline{E}^3)]_r &= i \frac{4\pi}{\omega} \left[\frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial}{\partial r}(r j_{3r}) \right) + ik \frac{\partial j_{3z}}{\partial r} \right] \\ [\nabla(\nabla \cdot \underline{E}^3)]_\phi &= 0 \\ [\nabla(\nabla \cdot \underline{E}^3)]_z &= - \frac{4\pi k}{\omega} \frac{1}{r} \frac{\partial}{\partial r}(r j_{3r}) - ik^2 j_{3z} \end{aligned}\right\} \text{(A1.4)}$$

The components of $\nabla^2 \underline{E}^3$ are given by

$$\left.\begin{aligned}
[\nabla^2 \underline{E}^3]_r &= \frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}(rE_r^3)\right) - k^2 E_r^3 \\
[\nabla^2 \underline{E}^3]_\phi &= \frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}(rE_\phi^3)\right) - k^2 E_\phi^3 \\
[\nabla^2 \underline{E}^3]_z &= \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial E_z^3}{\partial r}\right) - k^2 E_z^3
\end{aligned}\right\} \quad (A1.5)$$

Substituting equations (A1.4) and (A1.5) into equations (A1.2) results in equations (2.24). If k is set to zero, then equations (3.40) - (3.42) result.

APPENDIX 2 THE FUNCTION F_{imn}

$$F_{imn} = \frac{1}{s_{1i}^{6}}\left[I_{i1} + m_r s_{1i}^{2} I_{i2}\right]^{2} \frac{J_1^{2}(r\lambda_i)}{J_0^{4}(r_o\lambda_i)} \qquad \text{...(A2.1)}$$

It will now be shown that, $\lim_{i\to\infty} F_{imn} = 0$, for any m and n. Consider the following

$$\sum_{i=1}^{\infty}[F_{imn}]^{\frac{1}{2}} = \sum_{i=1}^{\infty}\frac{I_{i1}}{s_{1i}^{3}}\frac{J_1(r\lambda_i)}{J_0^{2}(r_o\lambda_i)} + m_r\sum_{i=1}^{\infty}\frac{I_{i2}}{s_{1i}}\frac{J_1(r\lambda_i)}{J_0^{2}(r_o\lambda_i)} \qquad \text{...(A2.2)}$$

Now

$$\sum_{i=1}^{\infty}\frac{I_{i1}}{s_{1i}^{3}}\frac{J_1(r\lambda_i)}{J_0^{2}(r_o\lambda_i)} = \frac{1}{r_o^{2}}\sum_{i=1}^{\infty}\frac{J_1(r\lambda_i)}{\lambda_i^{3}J_0^{2}(r_o\lambda_i)}\int_0^{r_o} rJ_1(r\lambda_i)G(r)\,dr \qquad \text{...(A2.3)}$$

where

$$G(r) = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}[J_1(k_{11}r)J_1(k_{12}r)]\right)\right] \qquad \text{...(A2.4)}$$

and

$$\sum_{i=1}^{\infty}\frac{I_{i2}}{s_{1i}}\frac{J_1(r\lambda_i)}{J_0^{2}(r_o\lambda_i)} = \frac{1}{r_o^{2}}\sum_{i=1}^{\infty}\frac{J_1(r\lambda_i)}{\lambda_i J_0^{2}(r_o\lambda_i)}\int_0^{r_o} rJ_1(r\lambda_i)H(r)\,dr \qquad \text{...(A2.5)}$$

where

$$H(r) = \frac{\partial}{\partial r}[J_1(k_{11}r)J_1(k_{12}r)] \qquad \text{...(A2.6)}$$

Since G(r) and H(r) both satisfy Dirichlet's conditions[†] in the closed interval $[0,r_o]$, for any finite

value of m and n (recall that $k_{11} = s_{1m}/r_o$, and $k_{12} = s_{1n}/r_o$), then

$$\frac{1}{r_o^2}\sum_{i=1}^{\infty} \frac{J_1(r\lambda_i)}{J_0^2(r_o\lambda_i)} \int_0^{r_o} rJ_1(r\lambda_i)G(r)dr, \text{ and}$$

$$\frac{1}{r_o^2}\sum_{i=1}^{\infty} \frac{J_1(r\lambda_i)}{J_0^2(r_o\lambda_i)} \int_0^{r_o} rJ_1(r\lambda_i)H(r)dr, \text{ are convergent (13).}$$

Hence, by comparison, the series in equations (A2.3) and (A2.5) are convergent. Therefore

$$\sum_{i=1}^{\infty} [F_{imn}] \quad \text{is bounded,} \quad => \quad \lim_{i\to\infty} F_{imn} = 0$$

A check of Tables A2.5 - A2.8 will verify that the convergence is rapid and that the best choice for i such that F_{imn} is maximized is i = 1.

Values of I_{i1} and $m_r s_{1i}^2 I_{i2}$ are given in tables A2.1 - A2.4. The integrals were solved numerically on an IBM 7040 computer by dividing the integrands into fifty segments and applying Simpson's rule. Those numbers which are of questionable accuracy due to computer roundoff error and error inherent in Simpson's rule are separated from the other numbers. The integrals are given by

$$I_{i1} = \int_0^1 uJ_1(s_{1i}u)\frac{\partial}{\partial u}\left[\frac{1}{u}\frac{\partial}{\partial u}\left(u\frac{\partial}{\partial u}[J_1(s_{1m}u)J_1(s_{1n}u)]\right)\right]du \qquad \text{...(A2.7)}$$

and

$$I_{i2} = \int_0^1 uJ_1(s_{1i}u)\frac{\partial}{\partial u}[J_1(s_{1m}u)J_1(s_{1n}u)]du \qquad \text{...(A2.8)}$$

To facilitate a computer solution, the derivative terms were expanded.

$$\frac{\partial}{\partial u}\left[\frac{1}{u}\,\frac{\partial}{\partial u}\left(u\frac{\partial}{\partial u}[J_1(s_{1m}u)J_1(s_{1n}u)]\right)\right]$$

$$\begin{aligned} = \quad & s_{1m}[8-(3s_{1n}^2+s_{1m}^2)u^2]J_0(s_{1m}u)J_1(s_{1n}u) \\ & + s_{1n}[8-(3s_{1m}^2+s_{1n}^2)u^2]J_0(s_{1n}u)J_1(s_{1m}u) \\ & - 4s_{1m}s_{1n}J_0(s_{1m}u)J_0(s_{1n}u) \\ & - 4[4-(s_{1m}^2+s_{1n}^2)u^2]\frac{J_1(s_{1m}u)J_1(s_{1n}u)}{u} \end{aligned}$$

...(A2.9)

$$\frac{\partial}{\partial u}[J_1(s_{1m}u)J_1(s_{1n}u)]$$

$$\begin{aligned} = \quad & s_{1m}J_0(s_{1m}u)J_1(s_{1n}u) \\ & + s_{1n}J_0(s_{1n}u)J_1(s_{1m}u) \\ & - 2\,\frac{J_1(s_{1m}u)J_1(s_{1n}u)}{u} \end{aligned}$$

...(A2.10)

Note that equations (A2.9) and (A2.10) verify that G(r) and H(r) satisfy Dirichlet's conditions.

[†] A function f(x) satisfies Dirichlet's conditions in the interval [a,b] if (13):

1. f(x) has only a finite number of maxima and minima in [a,b].
2. f(x) has only a finite number of finite discontinuities in [a,b] - and no infinite discontinuities.

TABLE A2.1 - VALUES OF I_{11} AND $S_{11}^2 m_r I_{12}$, l= 1

M/N	1	2	3	4	5	6	7	8	9	10
1		-0.154E 01 -0.606E-03	-0.919E 00 -0.873E-05	-0.104E 01 0.107E-05	-0.158E 01 -0.267E-06	-0.190E 01 0.857E-07	-0.227E 01 -0.402E-07	-0.259E 01 0.117E-07	-0.293E 01 -0.137E-07	-0.326E 01 -0.257E-08
2	-0.154E 01 -0.606E-03		-0.275E 01 -0.393E-03	-0.199E 01 -0.614E-05	-0.265E 01 0.785E-06	-0.350E 01 -0.210E-06	-0.411E 01 0.642E-07	-0.476E 01 -0.387E-07	-0.535E 01 0.350E-08	-0.597E 01 -0.184E-07
3	-0.919E 00 -0.873E-05	-0.275E 01 -0.393E-03		-0.457E 01 -0.290E-03	-0.365E 01 -0.471E-05	-0.479E 01 0.609E-06	-0.594E 01 -0.174E-06	-0.684E 01 0.450E-07	-0.776E 01 -0.394E-07	-0.863E 01 -0.662E-08
4	-0.104E 01 0.107E-05	-0.199E 01 -0.614E-05	-0.457E 01 -0.290E-03		-0.696E 01 -0.229E-03	-0.586E 01 -0.382E-05	-0.746E 01 0.492E-06	-0.891E 01 -0.152E-06	-0.101E 02 0.278E-07	-0.113E 02 -0.435E-07
5	-0.158E 01 -0.267E-06	-0.265E 01 0.785E-06	-0.365E 01 -0.471E-05	-0.696E 01 -0.229E-03		-0.990E 01 -0.190E-03	-0.861E 01 -0.322E-05	-0.107E 02 0.407E-06	-0.124E 02 -0.140E-06	-0.139E 02 0.110E-07
6	-0.190E 01 0.857E-07	-0.350E 01 -0.210E-06	-0.479E 01 0.609E-06	-0.586E 01 -0.382E-05	-0.990E 01 -0.190E-03		-0.134E 02 -0.162E-03	-0.119E 02 -0.278E-05	-0.144E 02 0.340E-06	-0.164E 02 -0.136E-06
7	-0.227E 01 -0.402E-07	-0.411E 01 0.642E-07	-0.594E 01 -0.174E-06	-0.746E 01 0.492E-06	-0.861E 01 -0.322E-05	-0.134E 02 -0.162E-03		-0.174E 02 -0.141E-03	-0.157E 02 -0.246E-05	-0.186E 02 0.284E-06
8	-0.259E 01 0.117E-07	-0.476E 01 -0.387E-07	-0.684E 01 0.450E-07	-0.891E 01 -0.152E-06	-0.107E 02 0.407E-06	-0.119E 02 -0.278E-05	-0.174E 02 -0.141E-03		-0.219E 02 -0.125E-03	-0.201E 02 -0.221E-05
9	-0.293E 01 -0.137E-07	-0.535E 01 0.350E-08	-0.776E 01 -0.394E-07	-0.101E 02 0.278E-07	-0.124E 02 -0.140E-06	-0.144E 02 0.340E-06	-0.157E 02 -0.246E-05	-0.219E 02 -0.125E-03		-0.270E 02 -0.112E-03
10	-0.326E 01 -0.257E-08	-0.597E 01 -0.184E-07	-0.863E 01 -0.662E-08	-0.113E 02 -0.435E-07	-0.139E 02 0.110E-07	-0.164E 02 -0.136E-06	-0.186E 02 0.284E-06	-0.201E 02 -0.221E-05	-0.270E 02 -0.112E-03	

TABLE A2.2 - VALUES OF I_{ij} AND $S_{12}^{2} m_r l_{12}$ $l= 2$

M/N	1	2	3	4	5	6	7	8	9	10
1		0.114E 01 -0.698E-03	-0.110E 01 -0.221E-02	-0.190E 01 -0.465E-04	-0.112E 01 0.686E-05	-0.193E 01 -0.194E-05	-0.215E 01 0.687E-06	-0.258E 01 -0.341E-06	-0.288E 01 0.118E-06	-0.324E 01 -0.120E-06
2	0.114E 01 -0.698E-03		-0.244E 01 -0.943E-03	-0.258E 01 -0.152E-02	-0.301E 01 -0.340E-04	-0.300E 01 0.516E-05	-0.407E 01 -0.157E-05	-0.460E 01 0.535E-06	-0.532E 01 -0.330E-06	-0.589E 01 0.597E-07
3	-0.110E 01 -0.221E-02	-0.244E 01 -0.943E-03		-0.489E 01 -0.772E-03	-0.457E 01 -0.116E-02	-0.479E 01 -0.267E-04	-0.540E 01 0.409E-05	-0.674E 01 -0.132E-05	-0.758E 01 0.393E-06	-0.857E 01 -0.332E-06
4	-0.190E 01 -0.465E-04	-0.258E 01 -0.152E-02	-0.489E 01 -0.772E-03		-0.768E 01 -0.635E-03	-0.706E 01 -0.940E-03	-0.716E 01 -0.220E-04	-0.833E 01 0.334E-05	-0.995E 01 -0.116E-05	-0.111E 02 0.266E-06
5	-0.112E 01 0.686E-05	-0.301E 01 -0.340E-04	-0.457E 01 -0.116E-02	-0.768E 01 -0.635E-03		-0.110E 02 -0.535E-03	-0.101E 02 -0.789E-03	-0.101E 02 -0.187E-04	-0.118E 02 0.279E-05	-0.137E 02 -0.107E-05
6	-0.193E 01 -0.194E-05	-0.300E 01 0.518E-05	-0.479E 01 -0.267E-04	-0.708E 01 -0.940E-03	-0.110E 02 -0.535E-03		-0.147E 02 -0.461E-03	-0.137E 02 -0.679E-03	-0.136E 02 -0.164E-04	-0.157E 02 0.235E-05
7	-0.215E 01 0.687E-06	-0.407E 01 -0.157E-05	-0.540E 01 0.409E-05	-0.716E 01 -0.220E-04	-0.101E 02 -0.789E-03	-0.147E 02 -0.461E-03		-0.190E 02 -0.404E-03	-0.178E 02 -0.597E-03	-0.176E 02 -0.146E-04
8	-0.258E 01 -0.341E-06	-0.460E 01 0.535E-06	-0.674E 01 -0.132E-05	-0.833E 01 0.334E-05	-0.101E 02 -0.187E-04	-0.137E 02 -0.679E-03	-0.190E 02 -0.404E-03		-0.238E 02 -0.360E-03	-0.224E 02 -0.532E-03
9	-0.288E 01 0.118E-06	-0.532E 01 -0.330E-06	-0.758E 01 0.393E-06	-0.995E 01 -0.116E-05	-0.118E 02 0.279E-05	-0.136E 02 -0.164E-04	-0.178E 02 -0.597E-03	-0.238E 02 -0.360E-03		-0.292E 02 -0.324E-03
10	-0.324E 01 -0.120E-06	-0.589E 01 0.597E-07	-0.857E 01 -0.332E-06	-0.111E 02 0.266E-06	-0.137E 02 -0.107E-05	-0.157E 02 0.235E-05	-0.176E 02 -0.146E-04	-0.224E 02 -0.532E-03	-0.292E 02 -0.324E-03	

TABLE A2.3 - VALUES OF I_{11} AND $s_{12}^{2} m_r I_{12}$ $l = 3$

M/N	1	2	3	4	5	6	7	8	9	10
1		-0.513E 01 0.560E-02	0.213E 01 -0.986E-03	-0.306E 00 -0.479E-02	-0.325E 01 -0.119E-03	-0.118E 01 0.192E-04	-0.231E 01 -0.583E-05	-0.239E 01 0.217E-05	-0.292E 01 -0.114E-05	-0.316E 01 0.406E-06
2	-0.513E 01 0.560E-02		-0.343E 01 -0.893E-03	-0.255E 01 -0.182E-02	-0.231E 01 -0.343E-02	-0.436E 01 -0.888E-04	-0.334E 01 0.148E-04	-0.469E 01 -0.483E-05	-0.510E 01 0.171E-05	-0.590E 01 -0.113E-05
3	0.213E 01 -0.986E-03	-0.343E 01 -0.893E-03		-0.529E 01 -0.112E-02	-0.525E 01 -0.158E-02	-0.465E 01 -0.267E-02	-0.624E 01 -0.708E-04	-0.802E 01 0.118E-04	-0.761E 01 -0.413E-05	-0.834E 01 0.126E-05
4	-0.306E 00 -0.479E-02	-0.255E 01 -0.182E-02	-0.529E 01 -0.112E-02		-0.778E 01 -0.103E-02	-0.821E 01 -0.134E-02	-0.746E 01 -0.219E-02	-0.877E 01 -0.590E-04	-0.922E 01 0.969E-05	-0.111E 02 -0.369E-05
5	-0.325E 01 -0.119E-03	-0.231E 01 -0.343E-02	-0.525E 01 -0.158E-02	-0.778E 01 -0.103E-02		-0.108E 02 -0.904E-03	-0.116E 02 -0.116E-02	-0.108E 02 -0.186E-02	-0.119E 02 -0.507E-04	-0.129E 02 0.810E-05
6	-0.118E 01 0.192E-04	-0.436E 01 -0.888E-04	-0.465E 01 -0.267E-02	-0.821E 01 -0.134E-02	-0.108E 02 -0.904E-03		-0.144E 02 -0.797E-03	-0.155E 02 -0.101E-02	-0.146E 02 -0.161E-02	-0.156E 02 -0.447E-04
7	-0.231E 01 -0.583E-05	-0.334E 01 0.148E-04	-0.624E 01 -0.708E-04	-0.746E 01 -0.219E-02	-0.116E 02 -0.116E-02	-0.144E 02 -0.797E-03		-0.186E 02 -0.709E-03	-0.199E 02 -0.896E-03	-0.189E 02 -0.143E-02
8	-0.239E 01 0.217E-05	-0.469E 01 -0.483E-05	-0.602E 01 0.118E-04	-0.877E 01 -0.590E-04	-0.108E 02 -0.186E-02	-0.155E 02 -0.101E-02	-0.186E 02 -0.709E-03		-0.233E 02 -0.637E-03	-0.249E 02 -0.804E-03
9	-0.292E 01 -0.114E-05	-0.510E 01 0.171E-05	-0.761E 01 -0.413E-05	-0.922E 01 0.969E-05	-0.119E 02 -0.507E-04	-0.146E 02 -0.161E-02	-0.199E 02 -0.896E-03	-0.233E 02 -0.637E-03		-0.285E 02 -0.578E-03
10	-0.316E 01 0.406E-06	-0.590E 01 -0.113E-05	-0.834E 01 0.126E-05	-0.111E 02 -0.369E-05	-0.129E 02 0.810E-05	-0.156E 02 -0.447E-04	-0.169E 02 -0.143E-02	-0.249E 02 -0.804E-03	-0.285E 02 -0.578E-03	

TABLE A2.4 - VALUES OF I_{ij} AND $s_{11}^{2} m_r I_{12}$ $l = 4$

M/N	1	2	3	4	5	6	7	8	9	10
1		-0.220E 00	-0.770E 01	0.307E 01	0.938E 00	-0.492E 01	-0.119E 01	-0.272E 01	-0.264E 01	-0.326E 01
		0.481E-03	0.943E-02	-0.128E-02	-0.837E-02	-0.226E-03	0.386E-04	-0.123E-04	0.470E-05	-0.263E-05
2	-0.220E 00		0.155E 01	-0.401E 01	-0.289E 01	-0.176E 01	-0.598E 01	-0.365E 01	-0.535E 01	-0.561E 01
	0.481E-03		0.108E-02	-0.114E-02	-0.296E-02	-0.611E-02	-0.172E-03	0.301E-04	-0.104E-04	0.372E-05
3	-0.770E 01	0.155E 01		-0.360E 01	-0.599E 01	-0.587E 01	-0.456E 01	-0.793E 01	-0.663E 01	-0.851E 01
	0.943E-02	0.108E-02		-0.129E-02	-0.172E-02	-0.268E-02	-0.483E-02	-0.139E-03	0.242E-04	-0.905E-05
4	0.307E 01	-0.401E 01	-0.360E 01		-0.657E 01	-0.866E 01	-0.904E 01	-0.775E 01	-0.106E 02	-0.101E 02
	-0.128E-02	-0.114E-02	-0.129E-02		-0.151E-02	-0.166E-02	-0.232E-02	-0.401E-02	-0.117E-03	0.200E-04
5	0.938E 00	-0.289E 01	-0.599E 01	-0.657E 01		-0.979E 01	-0.119E 02	-0.126E 02	-0.114E 02	-0.139E 02
	-0.837E-02	-0.296E-02	-0.172E-02	-0.151E-02		-0.143E-02	-0.150E-02	-0.203E-02	-0.343E-02	-0.101E-03
6	-0.492E 01	-0.176E 01	-0.587E 01	-0.866E 01	-0.979E 01		-0.134E 02	-0.157E 02	-0.167E 02	-0.155E 02
	-0.226E-03	-0.611E-02	-0.268E-02	-0.166E-02	-0.143E-02		-0.131E-02	-0.135E-02	-0.179E-02	-0.300E-02
7	-0.119E 01	-0.598E 01	-0.456E 01	-0.904E 01	-0.119E 02	-0.134E 02		-0.176E 02	-0.200E 02	-0.213E 02
	0.386E-04	-0.172E-03	-0.483E-02	-0.232E-02	-0.150E-02	-0.131E-02		-0.118E-02	-0.122E-02	-0.160E-02
8	-0.272E 01	-0.365E 01	-0.793E 01	-0.775E 01	-0.126E 02	-0.157E 02	-0.176E 02		-0.222E 02	-0.249E 02
	-0.123E-04	0.301E-04	-0.139E-03	-0.401E-02	-0.203E-02	-0.135E-02	-0.118E-02		-0.106E-02	-0.111E-02
9	-0.264E 01	-0.535E 01	-0.663E 01	-0.106E 02	-0.114E 02	-0.167E 02	-0.200E 02	-0.222E 02		-0.274E 02
	0.470E-05	-0.104E-04	0.242E-04	-0.117E-03	-0.343E-02	-0.179E-02	-0.122E-02	-0.106E-02		-0.985E-03
10	-0.326E 01	-0.561E 01	-0.851E 01	-0.101E 02	-0.139E 02	-0.155E 02	-0.213E 02	-0.249E 02	-0.274E 02	
	-0.263E-05	0.372E-05	-0.905E-05	0.200E-04	-0.101E-03	-0.300E-02	-0.160E-02	-0.111E-02	-0.985E-03	

TABLE A2.5 - $F_{\ell mn}$ $\ell = 1$

M/N	1	2	3	4	5	6	7	8	9	10
1		0.474E-02	0.169E-02	0.217E-02	0.503E-02	0.722E-02	0.103E-01	0.135E-01	0.172E-01	0.212E-01
2	0.474E-02		0.151E-01	0.791E-02	0.141E-01	0.245E-01	0.337E-01	0.453E-01	0.573E-01	0.712E-01
3	0.169E-02	0.151E-01		0.419E-01	0.266E-01	0.459E-01	0.705E-01	0.935E-01	0.120E 00	0.149E 00
4	0.217E-02	0.791E-02	0.419E-01		0.970E-01	0.686E-01	0.111E 00	0.159E 00	0.204E 00	0.255E 00
5	0.503E-02	0.141E-01	0.266E-01	0.970E-01		0.196E 00	0.148E 00	0.227E 00	0.308E 00	0.385E 00
6	0.722E-02	0.245E-01	0.459E-01	0.686E-01	0.196E 00		0.357E 00	0.283E 00	0.413E 00	0.540E 00
7	0.103E-01	0.337E-01	0.705E-01	0.111E 00	0.148E 00	0.357E 00		0.603E 00	0.494E 00	0.693E 00
8	0.135E-01	0.453E-01	0.935E-01	0.159E 00	0.227E 00	0.283E 00	0.603E 00		0.959E 00	0.805E 00
9	0.172E-01	0.573E-01	0.120E 00	0.204E 00	0.308E 00	0.413E 00	0.494E 00	0.959E 00		0.145E 01
10	0.212E-01	0.712E-01	0.149E 00	0.255E 00	0.385E 00	0.540E 00	0.693E 00	0.805E 00	0.145E 01	

TABLE A2.6 - $F_{\ell mn}$ $\ell = 2$

M/N	1	2	3	4	5	6	7	8	9	10
1		0.159E-03	0.149E-03	0.442E-03	0.154E-03	0.453E-03	0.564E-03	0.817E-03	0.101E-02	0.129E-02
2	0.159E-03		0.726E-03	0.813E-03	0.111E-02	0.110E-02	0.202E-02	0.259E-02	0.346E-02	0.424E-02
3	0.149E-03	0.726E-03		0.292E-02	0.255E-02	0.280E-02	0.357E-02	0.556E-02	0.703E-02	0.898E-02
4	0.442E-03	0.813E-03	0.292E-02		0.722E-02	0.614E-02	0.627E-02	0.847E-02	0.121E-01	0.150E-01
5	0.154E-03	0.111E-02	0.255E-02	0.722E-02		0.147E-01	0.125E-01	0.125E-01	0.169E-01	0.229E-01
6	0.453E-03	0.110E-02	0.280E-02	0.614E-02	0.147E-01		0.265E-01	0.230E-01	0.226E-01	0.303E-01
7	0.564E-03	0.202E-02	0.357E-02	0.627E-02	0.125E-01	0.265E-01		0.442E-01	0.367E-01	0.380E-01
8	0.817E-03	0.259E-02	0.556E-02	0.847E-02	0.125E-01	0.230E-01	0.442E-01		0.694E-01	0.615E-01
9	0.101E-02	0.346E-02	0.703E-02	0.121E-01	0.169E-01	0.226E-01	0.367E-01	0.694E-01		0.104E 00
10	0.129E-02	0.424E-02	0.898E-02	0.150E-01	0.229E-01	0.303E-01	0.380E-01	0.615E-01	0.104E 00	

TABLE A2.7 - F_{lmn} $l = 3$

M/N	1	2	3	4	5	6	7	8	9	10
1		0.568E-03	0.981E-04	0.209E-05	0.228E-03	0.300E-04	0.115E-03	0.124E-03	0.184E-03	0.216E-03
2	0.568E-03		0.255E-03	0.141E-03	0.115E-03	0.410E-03	0.241E-03	0.476E-03	0.563E-03	0.753E-03
3	0.981E-04	0.255E-03		0.605E-03	0.595E-03	0.468E-03	0.840E-03	0.784E-03	0.125E-02	0.150E-02
4	0.209E-05	0.141E-03	0.605E-03		0.131E-02	0.146E-02	0.120E-02	0.166E-02	0.184E-02	0.264E-02
5	0.228E-03	0.115E-03	0.595E-03	0.131E-02		0.254E-02	0.292E-02	0.251E-02	0.306E-02	0.362E-02
6	0.300E-04	0.410E-03	0.468E-03	0.146E-02	0.254E-02		0.451E-02	0.521E-02	0.460E-02	0.525E-02
7	0.115E-03	0.241E-03	0.840E-03	0.120E-02	0.292E-02	0.451E-02		0.746E-02	0.860E-02	0.774E-02
8	0.124E-03	0.476E-03	0.784E-03	0.166E-02	0.251E-02	0.521E-02	0.746E-02		0.117E-01	0.134E-01
9	0.184E-03	0.563E-03	0.125E-02	0.184E-02	0.306E-02	0.460E-02	0.860E-02	0.117E-01		0.175E-01
10	0.216E-03	0.753E-03	0.150E-02	0.264E-02	0.362E-02	0.525E-02	0.774E-02	0.134E-01	0.175E-01	

TABLE A2.8 - F_{lmn} $l = 4$

M/N	1	2	3	4	5	6	7	8	9	10
1		0.295E-06	0.361E-03	0.576E-04	0.527E-05	0.148E-03	0.866E-05	0.452E-04	0.424E-04	0.647E-04
2	0.295E-06		0.146E-04	0.983E-04	0.511E-04	0.191E-04	0.218E-03	0.813E-04	0.175E-03	0.192E-03
3	0.361E-03	0.146E-04		0.789E-04	0.219E-03	0.210E-03	0.127E-03	0.383E-03	0.268E-03	0.442E-03
4	0.576E-04	0.983E-04	0.789E-04		0.264E-03	0.457E-03	0.499E-03	0.366E-03	0.684E-03	0.624E-03
5	0.527E-05	0.511E-04	0.219E-03	0.264E-03		0.584E-03	0.863E-03	0.976E-03	0.790E-03	0.118E-02
6	0.148E-03	0.191E-04	0.210E-03	0.457E-03	0.584E-03		0.110E-02	0.150E-02	0.171E-02	0.147E-02
7	0.866E-05	0.218E-03	0.127E-03	0.499E-03	0.863E-03	0.110E-02		0.169E-02	0.244E-02	0.278E-02
8	0.452E-04	0.813E-04	0.383E-03	0.366E-03	0.976E-03	0.150E-02	0.169E-02		0.302E-02	0.377E-02
9	0.424E-04	0.175E-03	0.268E-03	0.684E-03	0.790E-03	0.171E-02	0.244E-02	0.302E-02		0.458E-02
10	0.647E-04	0.192E-03	0.442E-03	0.624E-03	0.118E-02	0.147E-02	0.278E-02	0.377E-02	0.458E-02	

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